

### A Vertex-Fitting Algorithm for the BCD

I sketch an algorithm to find a vertex from information in the silicon vertex detector of the BCD, for zero magnetic field. This algorithm seems more straightforward than the one found in subroutine VERTEX of the Cernlib, which we have been using up till now in GEANT studies. It allows a direct relation between the errors on measured hits and the  $\chi^2$ , as is always desired, and it never requires inversion of a matrix bigger than  $3 \times 3$ . Further, the contribution to the  $\chi^2$  from each track is readily displayed.

Label the tracks by  $i$ , while each track is measured in detector planes at positions  $z_j^i$ , yielding measured coordinates  $x_j^i$  and  $y_j^i$  with measurement errors  $(\sigma_{x,j}^i)$  and  $(\sigma_{y,j}^i)$ , respectively. The errors could and should include estimates of the (momentum dependent) position uncertainty due to multiple scattering, as well as that due to the detector resolution itself.

Strictly, the above description holds only for the 'disks,' which have the silicon planes perpendicular to the  $z$  axis. For the 'barrels' we should use a rotated coordinate system, as detailed in the sec. III.

From the track information we desire to extract the location of the vertex,  $(x_V, y_V, z_V)$ , and the slopes,  $m_x^i$  and  $m_y^i$ , of the tracks that emanate from the vertex. Then the tracks could be written

$$x - x_V = m_x(z - z_V), \quad y - y_V = m_y(z - z_V).$$

If there are  $N$  tracks hitting  $n$  doubled-sided silicon planes, we have  $2nN$  measurements and only  $2N + 3$  unknown parameters. So even if the tracks strike only 2 planes, a fit is assured for any  $N \geq 2$ . We also want a measure of the goodness of fit of each track to the vertex, to aid in a procedure to search for secondary vertices.

We will use a  $\chi^2$  method. The  $\chi^2$  which relates the unknown parameters directly to the measured quantities is

$$\chi^2 = \sum_i \sum_j \frac{(x_j^i - x_V - m_x^i(z_j^i - z_V))^2}{(\sigma_{x,j}^i)^2} + \frac{(y_j^i - y_V - m_y^i(z_j^i - z_V))^2}{(\sigma_{y,j}^i)^2}.$$

The fit values for the unknowns,  $x_V$ ,  $y_V$ ,  $z_V$ ,  $m_x^i$  and  $m_y^i$ , will be found by setting to zero the partial derivatives of the  $\chi^2$  with respect to the unknowns. The errors

on the fitted parameters are obtained from the second derivatives of the  $\chi^2$ , as specified below.

If we follow the procedure sketched above we quickly get  $2N + 3$  coupled non-linear equations. However, a simple iterative procedure leads to a solution involving only linear equations of at most 3 coupled variables:

- Initially set the  $m_x^i$  and  $m_y^i$  to their values as found by fits to the slopes of each track individually. Then we get 3 linear equations for the vertex coordinate,  $(x_V, y_V, z_V)$ .
- With the vertex thus found, calculate new values for the slopes by minimizing the  $\chi^2$ .
- With the new estimates of the slopes, return to the first step and iterate the procedure until satisfied of convergence...

### I. Solving for the vertex using fixed slopes.

Take partial derivatives:

$$\frac{\partial \chi^2}{\partial x_V} = -2 \sum_{i,j} \frac{x_j^i - x_V - m_x^i (z_j^i - z_V)}{(\sigma_{x,j}^i)^2} \quad (1)$$

$$\frac{\partial \chi^2}{\partial y_V} = -2 \sum_{i,j} \frac{y_j^i - y_V - m_y^i (z_j^i - z_V)}{(\sigma_{y,j}^i)^2} \quad (2)$$

$$\frac{\partial \chi^2}{\partial z_V} = 2 \sum_{i,j} \frac{m_x^i (x_j^i - x_V - m_x^i (z_j^i - z_V))}{(\sigma_{x,j}^i)^2} + \frac{m_y^i (y_j^i - y_V - m_y^i (z_j^i - z_V))}{(\sigma_{y,j}^i)^2} \quad (3)$$

On setting (1)-(3) to zero we obtain the linear equations

$$A_{11}x_V + A_{12}y_V + A_{13}z_V = B_1$$

$$A_{21}x_V + A_{22}y_V + A_{23}z_V = B_2$$

$$A_{31}x_V + A_{32}y_V + A_{33}z_V = B_3$$

and hence

$$(x_V, y_V, z_V) = A^{-1}B,$$

where

$$\begin{aligned}
A_{11} &= \sum_{i,j} \frac{1}{(\sigma_{x,j}^i)^2} \\
A_{12} &= 0 \\
-A_{13} &= \sum_{i,j} \frac{m_x^i}{(\sigma_{x,j}^i)^2} \\
A_{21} &= 0 \\
A_{22} &= \sum_{i,j} \frac{1}{(\sigma_{y,j}^i)^2} \\
-A_{23} &= \sum_{i,j} \frac{m_y^i}{(\sigma_{y,j}^i)^2} \\
A_{31} &= A_{13} \\
A_{32} &= A_{23} \\
A_{33} &= \sum_{i,j} \frac{(m_x^i)^2}{(\sigma_{x,j}^i)^2} + \frac{(m_y^i)^2}{(\sigma_{y,j}^i)^2} \\
B_1 &= \sum_{i,j} \frac{x_j^i - m_x^i z_j^i}{(\sigma_{x,j}^i)^2} \\
B_2 &= \sum_{i,j} \frac{y_j^i - m_y^i z_j^i}{(\sigma_{y,j}^i)^2} \\
-B_3 &= \sum_{i,j} \frac{m_x^i (x_j^i - m_x^i z_j^i)}{(\sigma_{x,j}^i)^2} + \frac{m_y^i (y_j^i - m_y^i z_j^i)}{(\sigma_{y,j}^i)^2}
\end{aligned}$$

The ‘errors’ are obtained from the second derivatives:

$$\begin{aligned}
\frac{1}{(\sigma_{x_V})^2} &= \frac{1}{2} \frac{\partial^2 \chi^2}{\partial x_V^2} = A_{11} = \sum_{i,j} \frac{1}{(\sigma_{x,j}^i)^2} \\
\frac{1}{(\sigma_{x_V y_V})^2} &= \frac{1}{2} \frac{\partial^2 \chi^2}{\partial x_V \partial y_V} = A_{12} = 0
\end{aligned}$$

*etc.* These quantities have the significance that

$$P(x_V, y_V, z_V) \sim \exp \left( -\frac{(x_V - \bar{x}_V)^2}{2(\sigma_{x_V})^2} - \frac{(x_V - \bar{x}_V)(y_V - \bar{y}_V)}{(\sigma_{x_V y_V})^2} - \dots \right).$$

Typically we are more interested in the second moments, which are related by

$$\langle x_V^2 \rangle = A_{11}^{-1}, \quad \langle x_V y_V \rangle = A_{12}^{-1} \quad \dots$$

The inverse matrix,  $A^{-1}$ , is also called the covariance matrix, I believe.

## II. Solving for the slopes given the vertex.

Again we take derivatives:

$$\frac{\partial \chi^2}{\partial m_x^i} = -2 \sum_j \frac{(z_j^i - z_V)(x_j^i - x_V - m_x^i(z_j^i - z_V))}{(\sigma_{x,j}^i)^2} \quad (4)$$

$$\frac{\partial \chi^2}{\partial m_y^i} = -2 \sum_j \frac{(z_j^i - z_V)(y_j^i - y_V - m_y^i(z_j^i - z_V))}{(\sigma_{y,j}^i)^2} \quad (5)$$

Setting these to zero we find:

$$m_x^i = \frac{C_1^i}{D_1^i}, \quad m_y^i = \frac{C_2^i}{D_2^i}$$

where

$$\begin{aligned} C_1^i &= \sum_j \frac{(z_j^i - z_V)(x_j^i - x_V)}{(\sigma_{x,j}^i)^2} \\ D_1^i &= \sum_j \frac{(z_j^i - z_V)^2}{(\sigma_{x,j}^i)^2} \\ C_2^i &= \sum_j \frac{(z_j^i - z_V)(y_j^i - y_V)}{(\sigma_{y,j}^i)^2} \\ D_2^i &= \sum_j \frac{(z_j^i - z_V)^2}{(\sigma_{y,j}^i)^2} \end{aligned}$$

The errors on the slopes are then given by:

$$\begin{aligned} \frac{1}{(\sigma_{m_x^i})^2} &= \frac{1}{2} \frac{\partial^2 \chi^2}{\partial (m_x^i)^2} = D_1^i = \sum_j \frac{(z_j^i - z_V)^2}{(\sigma_{x,j}^i)^2} \\ \frac{1}{(\sigma_{m_y^i})^2} &= \frac{1}{2} \frac{\partial^2 \chi^2}{\partial (m_y^i)^2} = D_2^i = \sum_j \frac{(z_j^i - z_V)^2}{(\sigma_{y,j}^i)^2} \end{aligned}$$

### III. Formalism for barrel detectors.

The barrel detectors are oriented with the  $z$  (beam) axis parallel to the silicon planes. We denote the angle between the normal to the silicon plane and the  $x$  axis by  $\phi$ . Introducing the  $u$  and  $v$  axes, with  $u$  perpendicular to the detector surface, we have

$$\begin{aligned} u &= x \cos \phi + y \sin \phi \\ v &= -x \sin \phi + y \cos \phi, \end{aligned}$$

where  $\phi$  is the angle between the  $u$  and  $x$  axes. The detected hits along a track in the barrel detectors are then parametrized by

$$\begin{aligned} v - v_V &= m_v^i (u - u_V) \\ z - z_V &= m_z^i (u - u_V), \end{aligned}$$

where the coordinates  $u$  measures the distance from the beam line to the detector. The  $\chi^2$  summed over barrel tracks is then

$$\begin{aligned} \chi^2 &= \sum_{i,j} \frac{(v_j^i - v_V - m_v^i (u_j^i - u_V))^2}{(\sigma_{v,j}^i)^2} + \frac{(z_j^i - z_V - m_z^i (u_j^i - u_V))^2}{(\sigma_{z,j}^i)^2} \\ &= \sum_{i,j} \frac{(v_j^i + x_V \sin \phi - y_V \cos \phi - m_v^i (u_j^i - x_V \cos \phi - y_V \sin \phi))^2}{(\sigma_{v,j}^i)^2} + \\ &\quad \frac{(z_j^i - z_V - m_z^i (u_j^i - x_V \cos \phi - y_V \sin \phi))^2}{(\sigma_{v,j}^i)^2}. \end{aligned}$$

As before, we set the derivatives of the  $\chi^2$  with respect to the unknown vertex and slopes to zero. First, fixing the slopes we obtain the following contributions to the  $A$  matrix and  $B$  vector:

$$\begin{aligned} A_{11} &= \sum_{i,j} \frac{(\sin \phi + m_v^i \cos \phi)^2}{(\sigma_{v,j}^i)^2} + \frac{(m_z^i)^2 \cos^2 \phi}{(\sigma_{z,j}^i)^2} \\ A_{12} &= \sum_{i,j} \frac{(\sin \phi + m_v^i \cos \phi)(-\cos \phi + m_v^i \sin \phi)}{(\sigma_{v,j}^i)^2} + \frac{(m_z^i)^2 \cos \phi \sin \phi}{(\sigma_{z,j}^i)^2} \\ -A_{13} &= \sum_{i,j} \frac{m_z^i \cos \phi}{(\sigma_{z,j}^i)^2} \\ A_{22} &= \sum_{i,j} \frac{(-\cos \phi + m_v^i \sin \phi)^2}{(\sigma_{v,j}^i)^2} + \frac{(m_z^i)^2 \sin^2 \phi}{(\sigma_{z,j}^i)^2} \end{aligned}$$

$$\begin{aligned}
-A_{23} &= \sum_{i,j} \frac{m_z^i \sin \phi}{(\sigma_{z,j}^i)^2} \\
A_{33} &= \sum_{i,j} \frac{1}{(\sigma_{z,j}^i)^2} \\
-B_1 &= \sum_{i,j} \frac{(\sin \phi + m_v^i \cos \phi)(v_j^i - m_v^i u_j^i)}{(\sigma_{v,j}^i)^2} + \frac{m_z^i \cos \phi (z_j^i - m_z^i u_j^i)}{(\sigma_{z,j}^i)^2} \\
-B_2 &= \sum_{i,j} \frac{(-\cos \phi + m_v^i \sin \phi)(v_j^i - m_v^i u_j^i)}{(\sigma_{v,j}^i)^2} + \frac{m_z^i \sin \phi (z_j^i - m_z^i u_j^i)}{(\sigma_{z,j}^i)^2} \\
B_3 &= \sum_{i,j} \frac{z_j^i - m_z^i u_j^i}{(\sigma_{z,j}^i)^2}
\end{aligned}$$

These terms are to be added to those for the disk detectors; the vertex coordinates are given by  $A^{-1}B$  as before, *etc.*

Then on fixing the vertex, we obtain the following for the slopes:

$$m_v^i = \frac{C_v^i}{D_v^i} \quad m_z^i = \frac{C_z^i}{D_z^i},$$

where

$$\begin{aligned}
C_v^i &= \sum_j \frac{(u_j^i - u_V)(v_j^i - v_V)}{(\sigma_{v,j}^i)^2} \\
D_v^i &= \sum_j \frac{(u_j^i - u_V)^2}{(\sigma_{v,j}^i)^2} \\
C_z^i &= \sum_j \frac{(u_j^i - u_V)(z_j^i - z_V)}{(\sigma_{z,j}^i)^2} \\
D_z^i &= \sum_j \frac{(u_j^i - u_V)^2}{(\sigma_{z,j}^i)^2}
\end{aligned}$$

The errors on the slopes are

$$\frac{1}{(\sigma_{m_v^i})^2} = D_v^i, \quad \frac{1}{(\sigma_{m_z^i})^2} = D_z^i.$$

#### IV. Estimating the measurement errors on the hits.

For the vertex-finding algorithm to work well, the relative weights of the different tracks must be accurately estimated. The weights are established via the contribution to the overall  $\chi^2$  of the individual hits, as scaled by the errors ( $\sigma_{x,j}^i$ ) and ( $\sigma_{y,j}^i$ ) for disk hits, and ( $\sigma_{v,j}^i$ ) and ( $\sigma_{z,j}^i$ ) for barrel hits. These errors should include both the measurement accuracy of the silicon detectors and the effect of multiple scattering in upstream material.

For the measurement error we use  $\sigma = 0.3 \times (\text{pitch})$ , where the strip pitch is  $50 \mu\text{m}$  in the present BCD configuration.

For multiple scattering, note that the material upstream of a silicon detector resides in thin layers: either the beam pipe or other silicon detectors. Hence the angular kick,  $\theta$ , due to multiple scattering causes a transverse displacement  $\sigma_{\perp}$  from the ideal track of  $L\theta$ , where  $L$  is the distance between the hit in question and the scattering material. The r.m.s. multiple scattering angle in the gaussian approximation is

$$\theta_{\text{proj}} = \frac{0.0141}{\beta P[\text{GeV}/c]} \sqrt{\frac{t}{t_0}} \left( 1 + \frac{1}{9} \log_{10} \frac{t}{t_0} \right),$$

where  $t_0$  is the radiation length of the scattering material, and  $t$  is the thickness traversed by the track in that material. The logarithmic correction is to our advantage given the thinness of the beam pipe and the silicon detectors. [However, the standard GEANT multiple-scattering routines do not include this factor!]

Strictly, the transverse displacement due to multiple scattering should include a term  $t\theta/\sqrt{3}$  due to the accumulated transverse displacement while traversing thickness  $t$  of the scattering material. However, as the material in question is the silicon detectors or the beam pipe, this contribution never amounts to more than  $1 \mu\text{m}$  and has been ignored.

The transverse displacement,  $\sigma_{\perp} = L\theta$ , must then be projected onto the measurement axes of the silicon detectors. Writing the vector from the point of scattering to the detector hit as

$$\vec{L} = (L_x, L_y, L_z),$$

we have for the projections in disk detectors,

$$\sigma_x = \sigma_{\perp} \frac{\sqrt{L_x^2 + L_z^2}}{L_z},$$

$$\sigma_y = \sigma_{\perp} \frac{\sqrt{L_y^2 + L_z^2}}{L_z},$$

while in the barrel detectors,

$$\begin{aligned}\sigma_v &= \sigma_{\perp}, \\ \sigma_z &= \sigma_{\perp} \frac{L}{\sqrt{L_x^2 + L_y^2}}.\end{aligned}$$

The contributions to the error on a given hit from detector resolution and the various multiple scatterings are added in quadrature.