A mechanical model that exhibits a gravitational critical radius

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I. PROBLEM

A popular model at science museums (and also a science toy[1]) that illustrates how curvature can be associated with gravity consists of a surface of revolution $r=-k/z$ with $z<0$ about a vertical axis z. The curvature of the surface, combined with the vertical force of Earth's gravity, leads to an inward horizontal acceleration of kg/r^2 for a particle that slides freely on the surface in a circular, horizontal orbit.

Consider the motion of a particle that slides freely on an arbitrary surface of revolution, $r=r(z)\geqslant 0$, defined by a continuous and differentiable function on some interval of z. The surface may have a nonzero minimum radius R at which the slope dr/dz is infinite. Discuss the character of oscillations of the particle about circular orbits to deduce a condition that there be a critical radius $r_{\mathrm{crit}}>R$, below which the orbits are unstable. That is, the motion of a particle with $r<r_{\mathrm{crit}}$ rapidly leads to excursions to the minimum radius R, after which the particle falls off the surface.

Give one or more examples of analytic functions $r(z)$ that exhibit a critical radius as defined above. These examples provide a mechanical analogy as to how departures of gravitational curvature from that associated with a $1/r^2$ force can lead to a characteristic radius inside which all motion tends toward a singularity.

II. SOLUTION

We work in a cylindrical coordinate system (r,θ,z) with the z axis vertical. It suffices to consider a particle of unit mass.

In the absence of friction, there is no torque on a particle about the z axis, so the angular momentum component $J=r^2\dot{\theta}$ about that axis is a constant of the motion, where the dot $(\cdot)$ indicates differentiation with respect to time.

For motion on a surface of revolution $r=r(z)$, we have $\dot{r}=r'\dot{z}$, where the prime $(\prime)$ indicates differentiation with respect to z. Hence, the kinetic energy can be written

$$T=\tfrac{1}{2}(\dot{r}^2+r^2\dot{\theta}^2+\dot{z}^2)=\tfrac{1}{2}[\dot{z}^2(1+r'^2)+r^2\dot{\theta}^2]. \tag{1}$$

The potential energy is $V=gz$. Using Lagrange's method, the equation of motion associated with the z coordinate is

$$\ddot{z}(1+r'^2)+\dot{z}^2rr''=-g+\frac{Jr'}{r^3}. \tag{2}$$

For a circular orbit at radius r_0, we have

$$r_0^3=\frac{J^2r_0'}{g}. \tag{3}$$

We write $\dot{\theta}_0=\Omega$, so that $J=r_0^2\Omega$.

For a perturbation about this orbit of the form

$$z=z_0+\epsilon\sin\omega t, \tag{4}$$

we have, to order ϵ,

$$r(z)\approx r(z_0)+r'(z_0)(z-z_0)=r_0+\epsilon r_0'\sin\omega t, \tag{5}$$

$$r'\approx r_0'+\epsilon r_0''\sin\omega t, \tag{6}$$

$$\frac{1}{r^3}\approx\frac{1}{r_0^3}\left(1-3\epsilon\frac{r_0'}{r_0}\sin\omega t\right). \tag{7}$$

Inserting (4)–(7) into (2) and keeping terms only to order ϵ, we obtain

$$-\epsilon\omega^2(1+r_0'^2)\sin\omega t \approx -g+\frac{J^2}{r_0^3}\left(r_0'-3\epsilon\frac{r_0'^2}{r_0}\sin\omega t+\epsilon r_0''\sin\omega t\right). \tag{8}$$

From the zero'th-order terms we recover (3), and from the order-ϵ terms we find that

$$\omega^2=\Omega^2\frac{3r_0'^2-r_0r_0''}{1+r_0'^2}. \tag{9}$$

The orbit is unstable when $\omega^2<0$, i.e., when

$$r_0r_0''>3r_0'^2. \tag{10}$$

This condition has the interesting geometrical interpretation (noted by a referee) that the orbit is unstable wherever

$$(1/r^2)''<0, \tag{11}$$

i.e., where the function $1/r^2$ is concave inwards.

For example, if $r=-k/z$, then $1/r^2=z^2/k^2$ is concave outwards, $\omega^2=J^2/(k^2+r_0^4)$, and there is no regime of instability.

We give three examples of surfaces of revolution that satisfy condition (11).

First, the hyperboloid of revolution defined by

$$r^2-z^2=R^2, \tag{12}$$

where R is a constant. Here, $r_0'=z_0/r_0$, $r_0''=R^2/r_0^3$, and

$$\omega^2=\Omega^2\frac{3z_0^2-R^2}{2z_0^2+R^2}=\Omega^2\frac{3r_0^2-4R^2}{2r_0^2-R^2}. \tag{13}$$

The orbits are unstable for

$$z_0<\sqrt{3}R, \tag{14}$$

or equivalently, for

$$r_0<\frac{2\sqrt{3}}{3}R=1.1547R\equiv r_{\mathrm{crit}}. \tag{15}$$

As r_0 approaches R, the instability growth time approaches an orbital period.

Another example is the Gaussian surface of revolution,

$$r^2 = R^2 e^{z^2}, \tag{16}$$

which has a minimum radius R, and a critical radius $r_{\text{crit}} = R\sqrt[4]{e} = 1.28R$.

Our final example is the surface

$$r = -\frac{k}{z\sqrt{1-z^2}} \quad (-1<z<0), \tag{17}$$

which has a minimum radius of $R=2k$, approaches the surface $r=-k/z$ at large r (small z), and has a critical radius of $r_{\text{crit}} = 6k/\sqrt{5} = 1.34R$.

These examples arise in a $2+1$ geometry with curved space but flat time. As such, they are not fully analogous to black holes in $3+1$ geometry with both curved space and curved time. Still, they provide a glimpse as to how a particle in curved space–time can undergo considerably more complex motion than in flat space–time.

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[1]The Vortx(tm) Miniature Wishing Well, Divnick International, Inc., 321 S. Alexander Road, Miamisburg, OH 45342, http://www.divnick.com/

AWESTRUCK SCIENTISTS

The second feature of science is that it shows that the world is simple. Even many scientists do not appreciate that they are hewers of simplicity from complexity. They are often more deluded than those they aim to tell. Scientists are often overawed by the complexity of detecting simplicity. They look at the latest fundamental particle experiment, see that it involves a thousand kilograms of apparatus and a discernible percentage of a gross national product, and become thunderstruck. They see the complexity of the apparatus and the intensity of the effort needed to construct and operate it, and confuse that with the simplicity that the experiment, if successful, will expose. Some scientists are so awestruck that they even turn to religion! Others keep a cool head, and marvel not at an implied design but at the richness of simplicity.

P. W. Atkins, ''The Limitless Power of Science,'' in *Nature's Imagination—The Frontiers of Scientific Vision*, edited by John Cornwell (Oxford University Press, New York, 1995).