NEW PROBLEMS

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''New Problems'' solicits interesting and novel worked problems for use in undergraduate physics courses beyond the introductory level. We seek problems that convey the excitement and interest of current developments in physics and that are useful for teaching courses such as Classical Mechanics, Electricity and Magnetism, Statistical Mechanics and Thermodynamics, ''Modern'' Physics, and Quantum Mechanics. We challenge physicists everywhere to create problems that show how contemporary research in their various branches of physics uses the central unifying ideas of physics to advance physical understanding. We want these problems to become an important source of ideas and information for students of physics and their teachers. All submissions are peer-reviewed prior to publication. Send manuscripts directly to Christopher R. Gould, *Editor*.

Levitating beachballs

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I. THE PROBLEM

Discuss the vertical and horizontal stability of a beachball levitated by a vertical jet of air. This demonstration is familiar to denizens of science museums and hardware stores.

As a complete solution is difficult, you may restrict your discussion to a simplified example. Consider a beachball of radius a in a jet such that the Reynolds number is a few hundred. Then, to a good approximation the air flow is incompressible and laminar. Viscous drag can be ignored. The high-speed-drag coefficient C_D can be taken as 2.

The diverging (but divergence-free) flow from the jet can be modeled by noting that the momentum flux in the jet is $\rho_{\text{air}} A v^2$. If the jet has some effective cone angle, its area expands with height z as z^2. Thus, to conserve the momentum flux, $v_z(z) \propto 1/z$. The transverse profile of the jet, $v_z(r)$, may be taken as Gaussian (or parabolic) in radius r.

You may also make the unrealistic approximation that the air flow is unperturbed by the beachball.

II. SOLUTION

A. Brief discussion

The beachball is subject to three forces: gravity, high-speed drag, and pressure-gradient forces. Gravity is downwards. The drag force is in the direction of the flow velocity $\mathbf{v}$, hence upwards and away from the axis of the vertical jet. The pressure-gradient force points in the direction of lower pressure, i.e., the direction of higher v^2 according to Bernoulli's law; hence, downwards and toward the axis of the jet.

For levitation to occur, the high-speed drag force must be large enough to counteract gravity and the pressure-gradient force. If so, we anticipate that vertical motion in the vicinity of the equilibrium point is stable. The situation for horizontal motion is less clear, since the (large) drag force destabilizes the equilibrium.

In the model discussed below, we find that the equilibrium is stable against vertical perturbations if the jet is reasonably well collimated.

B. The model

We work in cylindrical coordinates (r,z) with the z axis vertical.

In this problem, we are concerned with the flow on or near the z axis, where $v_r \ll v_z$. While a first approximation will suffice for v_r, we need a second approximation for v_z. To second order, the dependence of v_z on radius r can be approximated by a Gaussian, or by a parabola:

$$v_z \approx \frac{A}{z} e^{-r^2/2\beta^2 z^2} \approx \frac{A}{z}\left(1 - \frac{r^2}{2\beta^2 z^2}\right), \tag{1}$$

corresponding to cone angle β for the jet. For what it's worth, the jet momentum is $J_z = \pi \rho_a A^2 \beta^2$ where ρ_a is the density of air.

We determine v_r by requiring that $\nabla \cdot \mathbf{v} = 0$, which holds assuming air to be incompressible. All we need is the resulting first approximation:

$$v_r \approx \frac{Ar}{2z^2}. \tag{2}$$

For the record, the next approximation to v_r (which follows most quickly from the parabolic approximation to v_z) is

$$v_r = \frac{Ar}{2z^2}\left(1 - \frac{3r^2}{4\beta^2 z^2}\right). \tag{3}$$

Our model is a simplified version of Schlichting's (1933) solution for the laminar flow from a circular nozzle:[1]

$$v_z = \frac{A}{z} \frac{1}{(1 + Br^2/z^2)^2} \approx \frac{A}{z}(1 - 2Br^2/z^2)^2, \tag{4}$$

$$v_r = \frac{Ar}{2z^2} \frac{1 - Br^2/z^2}{(1 + Br^2/z^2)^2} \approx \frac{Ar}{2z^2}(1 - 3Br^2/z^2)^2. \tag{5}$$

Our model agrees with the approximate form of Schlichting's on identifying $B = 1/4\beta^2$.

C. The three forces on the ball

1. Gravity

$$\mathbf{F}_g = -mg\hat{\mathbf{z}} = -\tfrac{4}{3}\pi a^3 \rho_b g \hat{\mathbf{z}}, \tag{6}$$

where ρ_b is the density of the ball.

2. High-speed drag

$$\mathbf{F}_{\text{drag}} = \frac{C_D}{2} \rho_a \pi a^2 v^2 \hat{\mathbf{v}}, \tag{7}$$

where ρ_a is the density of the air and velocity $\mathbf{v}$ is evaluated at the center of the ball using the forms (1)–(2). We take $C_D/2 = 1$ in our regime.

3. Pressure-gradient effects

We again suppose that the ball does not perturb the pressure distribution. The net force in some direction u on the ball due to the pressure variation can be calculated using a spherical coordinate system centered on the ball, with angle θ measured with respect to the u axis:

$$F_{\nabla P} = -a^2 \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi P(a,\theta,\phi)\cos\theta. \tag{8}$$

We ignore the dependence of P on ϕ and approximate

$$P(a,\theta,\phi) \approx P(0) + P'(0) a \cos\theta + \cdots, \tag{9}$$

where the derivative is with respect to $u = a\cos\theta$. Then

$$F_{\nabla P} = -\tfrac{4}{3}\pi a^3 P' + \cdots. \tag{10}$$

We relate pressure to velocity by Bernoulli's equation (ignoring the gravitational pressure difference),

$$P + \tfrac{1}{2}\rho_a v^2 = P_0, \tag{11}$$

where P_0 is the atmospheric pressure far from the jet. Hence,

$$F_{\nabla P,u} = \frac{2}{3}\pi a^3 \rho_a \frac{\partial v^2}{\partial u} + \cdots. \tag{12}$$

D. The equilibrium point

Clearly, an equilibrium point lies on the z axis above the jet, so we take $u = z$ and write

$$\begin{aligned} F_z(z) &= F_g + F_{\text{drag},z} + F_{\nabla P,z} \\ &= -\frac{4}{3}\pi a^3 \rho_b g + \pi a^2 \rho_a v^2 + \frac{2}{3}\pi a^3 \rho_a \frac{dv^2}{dz}. \end{aligned} \tag{13}$$

Along the axis, the flow is given by (1) as $v^2 = A^2/z^2$, so

$$F_z(z) = -\frac{4}{3}\pi a^3 \rho_b g + \frac{\pi a^2 A^2 \rho_a}{z^2} - \frac{4\pi a^3 A^2 \rho_a}{3z^3}. \tag{14}$$

The equilibrium height z_0 then satisfies

$$\frac{1}{z_0^2}\left(1 - \frac{4a}{3z_0}\right) = \frac{4a\rho_b g}{3A^2\rho_a}. \tag{15}$$

The left-hand side of Eq. (15) must be positive, so the equilibrium height z_0 must be greater than $4a/3$ above the jet nozzle. If the ball is too heavy, it just falls onto the nozzle in our approximation. In practice, as the ball comes very close to the nozzle, the perturbations in the flow cannot be ignored, and an equilibrium might still be possible.

E. Vertical oscillations about equilibrium

As usual, we expand

$$F_z(z) = F'(z_0)(z - z_0) + \cdots. \tag{16}$$

If $F'(z_0)$ is negative then the vertical motion is stable, with oscillations about equilibrium at frequency $\omega = \sqrt{-F'(z_0)/m}$. From (14) we find

$$F'(z_0) = -\frac{2\pi a^2 A^2 \rho_a}{z_0^3}\left(1 - \frac{2a}{z_0}\right). \tag{17}$$

It appears that the vertical equilibrium is stable only for $z_0 \gtrsim 2a$, a slightly stronger condition than (15) that the equilibrium exist.

F. Stability of horizontal motion

The horizontal force on the ball in the plane $z = z_0$ is

$$F_r(r,z_0) = F_{\text{drag},r} + F_{\nabla P,r} = \pi a^2 \rho_a v v_r + \frac{2}{3}\pi a^3 \rho_a \frac{\partial v^2}{\partial r}, \tag{18}$$

using (7) and (12). We are only interested in small perturbations away from the axis $r = 0$, so we keep terms in F_r only to order r/z, corresponding to order r^2/z^2 in v^2. Then,

$$v^2 = v_z^2 + v_r^2 \approx \frac{A^2}{z^2}\left(1 - \frac{r^2}{\beta^2 z^2} + \frac{r^2}{4z^2}\right), \tag{19}$$

$$\frac{\partial v^2}{\partial r} \approx -\frac{2A^2 r}{\beta^2 z^4}\left(1 - \frac{\beta^2}{4}\right), \tag{20}$$

from (1) and (2), and

$$v v_r \approx \frac{A^2 r}{2z^3}. \tag{21}$$

Combining (18)–(21),

$$F_r(r,z_0) \approx -\frac{4\pi a^3 A^2 \rho_a r}{3\beta^2 z_0^4}\left(1 - \frac{\beta^2}{4} - \frac{3\beta^2 z_0}{8a}\right). \tag{22}$$

In our model, β is the cone angle of the jet, so $\beta \lesssim 1$; otherwise we could hardly speak of a jet. Hence, the term $\beta^2/4$ will not cause instability by itself. The term $3\beta^2 z_0/8a$ could lead to instability if $z_0 \gg a$, i.e., for equilibrium points very far above the jet nozzle. Since βz_0 is the radius of the jet at height z_0, we must have $\beta z_0 \gtrsim a$ for the ball to be within the jet, so the model is meaningful. If we take $\beta z_0 \approx 2a$ as the working regime, then $3\beta^2 z_0/8a \approx 3\beta/4$. Then, (22) indicates that there would be horizontal stability for any value of β less than 1, i.e., for all reasonable jets.

The frequencies of the horizontal and vertical oscillations are typically not the same, and result in a ''jumpy'' appearance to the motion of a levitating beachball. For $z_0 \gg a$ and $\beta z_0 = ka$, the ratio of frequencies is

$$\frac{\omega_{\text{horiz}}}{\omega_{\text{vert}}} \approx \frac{1}{k}\sqrt{\frac{2z_0}{3a}}. \tag{23}$$

This can be close to unity, for example, with $z_0 = 6a$ and $\beta z_0 = 2a$.

Among the possibilities for the orbit of the horizontal oscillations is a circle.

[1]H. Schlichting, *Boundary Layer Theory* (McGraw–Hill, New York, 1979), 7th ed., Sec. XI.a.2.