NEW PROBLEMS

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''New Problems'' resumes this month with the first of a series of problems from Kirk McDonald of Princeton University. ''New Problems'' continues to solicit interesting and novel worked problems for use in undergraduate physics courses beyond the introductory level. We seek problems that convey the excitement and interest of current developments in physics and that are useful for teaching courses such as Classical Mechanics, Electricity and Magnetism, Statistical Mechanics and Thermodynamics, ''Modern'' Physics, and Quantum Mechanics. We challenge physicists everywhere to create problems that show how their various branches of physics use the central unifying ideas of physics to advance physical understanding. We want these problems to become an important source of ideas and information for students of physics and their teachers. All submissions are peer-reviewed prior to publication. Send manuscripts directly to Christopher R. Gould, *Editor*.

Slow light

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I. PROBLEM

Consider a classical model of matter in which spectral lines are associated with oscillators. In particular, consider a gas with two closely spaced spectral lines, $\omega_{1,2}=\omega_0\pm\Delta$, where $\Delta\ll\omega_0$. Each line has oscillator strength $1/Z$, where Z is the atomic number of a gas atom, and each has the same damping constant (and spectral width) γ. For simplicity, you may suppose that $\Delta=\gamma$.

Ordinarily, the gas would exhibit strong absorption of light in the vicinity of the spectral lines. But suppose that a laser of frequency ω_2 ''pumps'' the second oscillator into an inverted population. Classically, this is described by assigning a negative damping constant to this oscillator: $\gamma_2=-\gamma$.

Deduce an expression for the group velocity of a pulse of light centered on frequency ω_0 in this medium. Show also that frequencies very near ω_0 propagate without attenuation.

In a recent experiment,[1] the group velocity of light was reduced to 38 mph (17 m/s) by this technique in a sodium vapor of density $N=5\times10^{12}$ atoms/cm^3 using a pair of lines for which $2\Delta\approx10^7$/s.

II. SOLUTION

In a medium of index of refraction $n(\omega)$, the dispersion relation can be written

$$k=\frac{\omega n}{c}, \tag{1}$$

where k is the wave number and c is the speed of light. The group velocity is then given by

$$v_g=\frac{d\omega}{dk}=\frac{1}{dk/d\omega}=\frac{c}{n+\omega\dfrac{dn}{d\omega}}. \tag{2}$$

We next recall the classical oscillator model for the index of refraction. The index n is the square root of the dielectric constant ϵ, which is in turn related to the atomic polarizability α according to (in Gaussian units)

$$D=\epsilon E=E+4\pi P=E(1+4\pi N\alpha), \tag{3}$$

where D is the electric displacement, E is the electric field, P is the polarization density, and N is the atomic number density. Then,

$$n=\sqrt{\epsilon}\approx1+2\pi N\alpha, \tag{4}$$

for a dilute gas with index near 1.

The polarizability α is obtained from the dipole moment $p=ex=\alpha E$ induced by electric field E. In the case of a single spectral line of frequency ω_0, we say that the charge e is bound to the (fixed) nucleus by a spring of constant $k=m\omega_0^2$, and the motion is subject to damping $-m\gamma\dot{x}$. The equation of motion in the presence of a wave of frequency ω is

$$\ddot{x}+\gamma\dot{x}+\omega_0^2x=\frac{eE}{m}=\frac{eE_0}{m}e^{i\omega t}. \tag{5}$$

Hence,

$$x=\frac{eE}{m}\frac{1}{\omega_0^2-\omega^2+i\gamma\omega}=\frac{eE}{m}\frac{\omega_0^2-\omega^2-i\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}, \tag{6}$$

and so the polarizability is

$$\alpha=\frac{e^2}{m}\frac{\omega_0^2-\omega^2-i\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}. \tag{7}$$

In the present problem, we have two spectral lines, $\omega_{1,2}=\omega_0\mp\gamma$, both of oscillator strength $1/Z$, but the population of line 2 is inverted so that $\gamma_2=-\gamma_1=-\gamma$. In this case, the polarizability is given by

$$\alpha=\frac{1}{Z}\frac{e^2}{m}\frac{(\omega_0-\gamma)^2-\omega^2-i\gamma\omega}{((\omega_0-\gamma)^2-\omega^2)^2+\gamma^2\omega^2}+\frac{1}{Z}\frac{e^2}{m}\frac{(\omega_0+\gamma)^2-\omega^2+i\gamma\omega}{((\omega_0+\gamma)^2-\omega^2)^2+\gamma^2\omega^2}$$
$$\approx\frac{1}{Z}\frac{e^2}{m}\frac{\omega_0^2-2\gamma\omega_0-\omega^2-i\gamma\omega}{(\omega_0^2-2\gamma\omega_0-\omega^2)^2+\gamma^2\omega^2}+\frac{1}{Z}\frac{e^2}{m}\frac{\omega_0^2+2\gamma\omega_0-\omega^2+i\gamma\omega}{(\omega_0^2+2\gamma\omega_0-\omega^2)^2+\gamma^2\omega^2}, \tag{8}$$

where the approximation is obtained by the neglect of terms in γ^2 compared to those in $\gamma\omega_0$.

We now consider the issue of attenuation of a pulse of frequency ω. Since $k=\omega n/c\approx\omega(1+2\pi N\alpha)/c$, the spatial dependence e^{ikz} of a pulse propagating in the z direction includes attenuation if the imaginary part of the index n is nonzero. However, the population inversion described by $\gamma_2=-\gamma_1$ leads to $\mathrm{Im}[\alpha(\omega_0)]=0$. Hence, there is no attenuation of a probe pulse at frequency ω_0.

In the present model, the pulse is attenuated at frequencies less than ω_0, but grows (lases) at frequencies greater than ω_0. In the experiment of Hau *et al.*,[1] lasing did not occur because line 2 actually corresponded to a transition between the upper level of line 1 and a third, excited level. (In a sense, the quantum mechanical level structure with one high and two low energy levels is the inverse of that assumed in the classical model here, i.e., one low and two high levels.) Therefore, pumping at frequency ω_2 did not produce an inverted population that could lead to lasing; but it did lead to an effective sign reversal of the damping constant γ_2 for a narrow range of frequencies near ω_0.

To obtain the group velocity at frequency ω_0, we need the derivative

$$\left.\frac{d\,\mathrm{Re}(n)}{d\omega}\right|_{\omega_0}=2\pi N\left.\frac{d\,\mathrm{Re}(\alpha)}{d\omega}\right|_{\omega_0}=\frac{24\pi Ne^2}{25Zm\gamma^2\omega_0}. \tag{9}$$

Since $\alpha(\omega_0)=0$, we have $n(\omega_0)=1$, and the phase velocity at ω_0 is exactly c. The group velocity (2) is

$$v_g=\frac{c}{1+\dfrac{24\pi Ne^2}{25Zm\gamma^2}}\approx\frac{25Z\gamma^2}{24\pi N\dfrac{e^2}{mc^2}c^2}c\approx\frac{Z\gamma^2}{\pi Nr_0c}, \tag{10}$$

where $r_0=e^2/mc^2\approx3\times10^{-13}$ cm is the classical electron radius. The group velocity is lower in a denser medium.

In the experiment of Hau *et al.*, the medium was sodium vapor ($Z=11$), cooled to less than 1 μK to increase the density. An additional increase in density by a factor of 5 was obtained when the vapor formed a Bose condensate. Plugging in the experimental parameters, $N=5\times10^{12}/\mathrm{cm}^3$ and $\gamma=5\times10^6/\mathrm{s}$, we find

$$v_g\approx\frac{11\cdot(5\times10^6)^2}{3\cdot5\times10^{12}\cdot3\times10^{-13}\cdot3\times10^{10}}\approx2000\ \mathrm{cm/s}, \tag{11}$$

compared to the measured value of 1700 cm/s.

[1] L. V. Hau *et al.*, ''Light speed reduction to 17 metres per second in an ultracold atomic gas,'' Nature (London) **397**, 594–598 (1999).

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Edward O. Wilson, ''Scientists, Scholars, Knaves and Fools,'' Am. Scientist **86** (1), 6–7 (1998).