Circular orbits inside the sphere of death

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A wheel or sphere rolling without slipping on the inside of a sphere in a uniform gravitational field can have stable circular orbits that lie wholly above the ''equator,'' while a particle sliding freely cannot.

I. INTRODUCTION

In a recent article[1] in this Journal, Abramowicz and Szuszkiewicz remarked on an interesting analogy between orbits above the equator of a ''wall of death'' and orbits near a black hole; namely, that the centrifugal force in both cases appears to point toward, rather than away from, the center of an appropriate coordinate system. Here we take a wall of death to be a hollow sphere on the Earth's surface large enough that a motorcycle can be driven on the inside of the sphere. The intriguing question is whether there exist stable orbits for the motorcycle that lie entirely above the equator (horizontal great circle) of the sphere.

In Ref. 1 the authors stated that no such orbits are possible, perhaps recalling the well-known result for a particle sliding freely on the inside of a sphere in a uniform gravitational field. However, the extra degrees of freedom associated with a rolling wheel (or sphere) actually do permit such orbits, in apparent defiance of intuition. In particular, the friction associated with the condition of rolling without slipping can, in some circumstances, have an upward component large enough to balance all other downward forces.

In this paper we examine the character of all circular orbits inside a fixed sphere, for both wheels and spheres that roll without slipping. The rolling constraint is velocity dependent (nonholonomous), so explicit use of a Lagrangian is not especially effective. Instead, we follow a vectorial approach as advocated by Milne (Chap. 17).[2] This approach does utilize the rolling constraint, a careful choice of coordinates, and the elimination of the constraint force from the equations of motion, all of which are implicit in Lagrange's method. The vector approach is, of course, a convenient codification of earlier methods in which individual components were explicitly written out. Compare with classic works such as those of Lamb (Chap. 9),[3] Deimel (Chap. 7),[4] and Routh (Chap. 5).[5]

Once the solutions are obtained in Sec. II for rolling wheels we make a numerical evaluation of the magnitude of the acceleration in g's, and of the required coefficient of static friction on some representative orbits. The resulting parameters are rather extreme, and the circus name ''sphere of death'' seems apt.

The stability of steady orbits of wheels is considered in some detail, but completely general results are not obtained (because the general motion has four degrees of freedom). All vertical orbits are shown to be stable, as are horizontal orbits around the equator of the sphere. We also find that all horizontal orbits away from the poles are stable in the limit of small wheels, and conjecture that a similar condition holds for ''death-defying'' orbits of large wheels above the equator of the sphere. In Sec. IV we lend support to this conjecture by comparing the related case of a sphere rolling within a sphere for which a complete stability analysis can be given.

Discussions of wheels and spheres rolling outside a fixed sphere are given in Secs. III and V, respectively.

II. WHEEL ROLLING INSIDE A FIXED SPHERE

A. Generalities

We consider a wheel of radius a rolling without slipping on a circular orbit on the inner surface of a sphere of radius $r > a$. The analysis is performed in the lab frame, in which the sphere is fixed. The z axis is vertical and upwards with its origin at the center of the sphere, as shown in Fig. 1. As the wheel rolls on the sphere, the point of contact traces a path that is an arc of a circle during any short interval. In steady motion the path forms closed circular orbits which are of primary interest here. We therefore introduce a set of axes (x', y', z') that are related to the circular motion of the point of contact. If the motion is steady, these axes are fixed in the lab frame.

The normal to the plane of the circular orbit through the center of the sphere (and also through the center of the circle) is labeled z'. The angle between axes z and z' is β with $0 \leqslant \beta \leqslant \pi/2$. A radius from the center of the sphere to the point of contact of the wheel sweeps out a cone of angle θ about the z' axis, where $0 \leqslant \theta \leqslant \pi$. The azimuthal angle of the point of contact on this cone is called ϕ, with $\phi = 0$ defined by the direction of the x' axis, which is along the projection of the z axis onto the plane of the orbit, as shown in Fig. 2. Unit vectors are labeled with a superscript caret ($\hat{\ }$), so that $\hat{y}' = \hat{z}' \times \hat{x}'$ completes the definition of the prime-coordinate system.

For a particle sliding freely, the only stationary orbits have $\beta = 0$ (horizontal circles) or $\beta = \pi/2$ (vertical great circles). For wheels and spheres rolling inside a sphere there are orbits only with $\beta = 0$ or $\pi/2$ also, as we will demonstrate. However, the friction at the point of contact in the rolling cases permits orbits with a larger range of θ than in the sliding case. If $\beta = 0$ or $\pi/2$ were accepted as an assumption, the derivation could be shortened somewhat.

We also introduce a right-handed coordinate triad of unit vectors $(\hat{1}, \hat{2}, \hat{3})$ related to the geometry of the wheel. Axis $\hat{1}$ lies along the symmetry axis of the wheel, as shown in Fig. 1. Axis $\hat{3}$ is directed from the center of the wheel to the point of contact of the wheel with the sphere. The vector from the center of the wheel to the point of contact is then

$$\mathbf{a} = a\hat{3}. \tag{1}$$

Axis $\hat{2} = \hat{3} \times \hat{1}$ lies in the plane of the wheel, and also in the plane of the orbit (the $x' - y'$ plane). The sense of axis $\hat{1}$ is chosen so that the component ω_1 of the angular velocity vector $\vec{\omega}$ of the wheel about this axis is positive. Conse-

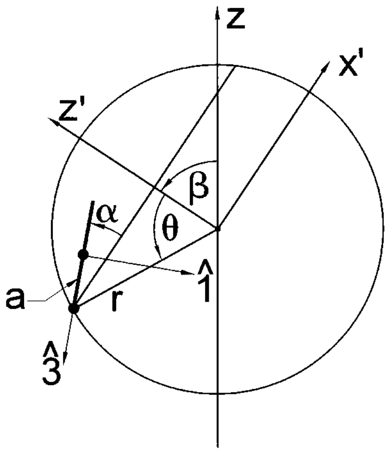

Fig. 1. A wheel of radius a rolls without slipping on a circular orbit inside a fixed sphere of radius r. The orbit sweeps out a cone of angle θ about the z' axis, which axis makes angle β to the vertical. The x' axis is orthogonal to the z' axis in the $z-z'$ plane. The angle between the plane of the orbit and diameter of the wheel that includes the point of contact with the sphere is denoted by α. A right-handed triad of unit vectors, $(\hat{1},\hat{2},\hat{3})$, is defined with $\hat{1}$ along the axis of the wheel and $\hat{3}$ pointing from the center of the wheel to the point of contact.

quently, axis $\hat{2}$ points in the direction of the velocity of the point of contact and therefore is parallel to the tangent to the orbit.

Except for axis $\hat{1}$, these rotating axes are not body axes, but the inertia tensor is diagonal with respect to them. We write

$$I_{11}=2kma^2, \quad I_{22}=kma^2=I_{33}, \tag{2}$$

which holds for any circularly symmetric disk according to the perpendicular axis theorem; $k=1/2$ for a wheel of radius a with mass m concentrated at the rim, $k=1/4$ for a uniform disk, etc.

The wheel does not necessarily lie in the plane of the orbit. Indeed, it is the freedom to ''bank'' the wheel that makes the ''death-defying'' orbits possible. The diameter of the wheel through the point of contact (i.e., axis $\hat{3}$) makes angle α to the plane of the orbit. In general, a wheel can have an arbitrary rotation about the $\hat{3}$ axis, but the wheel will roll steadily along a closed circular orbit orbit only if the angular velocity component ω_3 is such that the plane of the wheel intersects the plane of the orbit along the tangent to the orbit at the point of contact. Hence, for steady motion we will be able to deduce a constraint on ω_3. The case of a rolling sphere is distinguished by the absence of this constraint, as considered later.

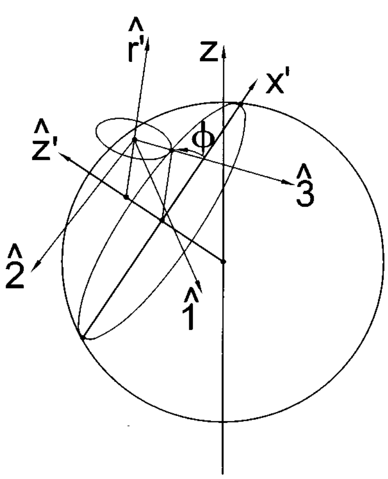

Fig. 2. The azimuth of the point of contact of the wheel with the sphere to the x' axis is ϕ. The unit vector $\hat{r}'$ is orthogonal to the z' axis and points toward the center of the wheel (or equivalently, toward the point of contact). Unit vector $\hat{2}=\hat{3}\times\hat{1}=\hat{z}'\times\hat{r}'$.

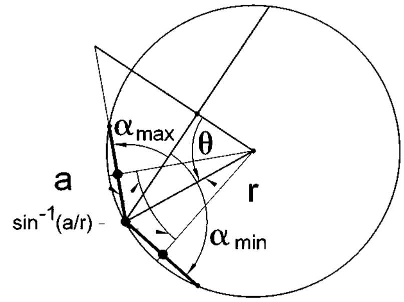

Fig. 3. Geometry illustrating the extremes of angle α.

Since the wheel lies inside the sphere as shown in Fig. 3, we can readily deduce the geometric relation that

$$\theta-\pi+\sin^{-1}(a/r)<\alpha<\theta-\sin^{-1}(a/r). \tag{3}$$

It is useful to introduce $\mathbf{r}'=r'\hat{r}'$ as the perpendicular vector from the z' axis to the center of the wheel. The magnitude r' is given by

$$r'=r\sin\theta-a\cos\alpha, \tag{4}$$

as shown in Fig. 4. The vector $\hat{z}'\times\hat{r}'$ is in the direction of motion of the point of contact, which was defined previously to be direction $\hat{2}$. That is, $(\hat{r}',\hat{2},\hat{z}')$ form a right-handed unit triad. The length r' is negative when the center of the wheel is on the opposite side of the z' axis from the point of contact. This can occur for large enough a/r when the point of contact is near the z' axis, such as when $\theta\approx0$ and $\alpha<0$ or $\theta\approx\pi$ and $\alpha>0$.

The force of contact of the sphere on the wheel is labeled **F**. For the wheel to be in contact with the sphere, the force **F** must have a component toward the center of the sphere, which will be verified after the motion is obtained.

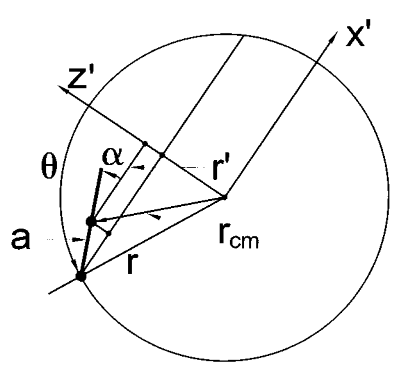

Fig. 4. Geometry illustrating the vector $\mathbf{r}_{\rm cm}$ from the center of the sphere to the center of the wheel, and the distance $r'=r\sin\theta-a\cos\alpha$ from the z' axis to the center of the wheel.

The equation of motion of the center of mass of the wheel is

$$m\,\frac{d^2\mathbf{r}_{\rm cm}}{dt^2}=\mathbf{F}-mg\hat{z}, \tag{5}$$

where g is the acceleration due to gravity. The equation of motion for rotation about the center of mass is

$$\frac{d\mathbf{L}_{\rm cm}}{dt}=\mathbf{N}_{\rm cm}=\mathbf{a}\times\mathbf{F}. \tag{6}$$

We eliminate the unknown force $\mathbf{F}$ in Eq. (6) via Eqs. (1) and (5) to find

$$\frac{1}{ma}\,\frac{d\mathbf{L}_{\rm cm}}{dt}=g\hat{3}\times\hat{z}+\hat{3}\times\frac{d^2\mathbf{r}_{\rm cm}}{dt^2}. \tag{7}$$

The constraint that the wheel rolls without slipping relates the velocity of the center of mass to the angular velocity vector $\boldsymbol{\omega}$ of the wheel. In particular, the velocity vanishes for that point on the wheel instantaneously in contact with the sphere:

$$\mathbf{v}_{\rm contact}=\mathbf{v}_{\rm cm}+\boldsymbol{\omega}\times\mathbf{a}=0, \tag{8}$$

and hence,

$$\mathbf{v}_{\rm cm}=\frac{d\mathbf{r}_{\rm cm}}{dt}=a\hat{3}\times\boldsymbol{\omega}. \tag{9}$$

Multiplying this equation by $\hat{3}$ we find

$$\boldsymbol{\omega}=-\hat{3}\times\frac{\mathbf{v}_{\rm cm}}{a}+\omega_3\hat{3}. \tag{10}$$

Equations (5)–(10) hold whether the rolling object is a wheel or a sphere.

The strategy now is to extract as much information as possible about the angular velocity $\boldsymbol{\omega}$ before confronting the full equation of motion (7). The angular velocity can also be written in terms of any unit body vector $\hat{1}$ fixed in the wheel as

$$\boldsymbol{\omega}=\omega_1\hat{1}+\hat{1}\times\frac{d\hat{1}}{dt}. \tag{11}$$

This follows on writing $\boldsymbol{\omega}=w_1\hat{1}+\boldsymbol{\omega}_\perp$ and noting that the rate of change of the body vector $\hat{1}$ is just $d\hat{1}/dt=\boldsymbol{\omega}_\perp\times\hat{1}$, so $\boldsymbol{\omega}_\perp=\hat{1}\times d\hat{1}/dt$. In particular, Eq. (11) holds for our choice of $\hat{1}$ as the symmetry axis of the wheel. Using Eq. (2) the angular momentum can now be written as

$$\mathbf{L}=\overleftrightarrow{\mathbf{I}}\cdot\boldsymbol{\omega}=2kma^2\omega_1\hat{1}+kma^2\hat{1}\times\frac{d\hat{1}}{dt}. \tag{12}$$

B. Steady motion in a circle

To obtain additional relations we restrict our attention to orbits in which the point of contact of the wheel with the sphere moves in a closed circle. In such cases the center of mass of the wheel [and also the coordinate triad $(\hat{1},\hat{2},\hat{3})$] has angular velocity $\dot{\phi}$ about the $\hat{z}'$ axis (and no other component), where the dot means differentiation with respect to time. Thus

$$\mathbf{v}_{\rm cm}=\dot{\phi}\hat{z}'\times r'\hat{r}'=r'\dot{\phi}\hat{2}. \tag{13}$$

Equation (10) can now be evaluated, yielding

$$\boldsymbol{\omega}=(r'/a)\dot{\phi}\hat{1}+\omega_3\hat{3}. \tag{14}$$

For steady motion there can be no rotation about axis $\hat{2}$; angle α is constant. To find ω_3 we now pursue Eq. (11).

As argued above, the angular velocity $\boldsymbol{\gamma}$ of the triad $(\hat{1},\hat{2},\hat{3})$ is

$$\boldsymbol{\gamma}=\dot{\phi}z'=-\dot{\phi}\cos\alpha\hat{1}-\dot{\phi}\sin\alpha\hat{3}, \tag{15}$$

using

$$\hat{z}'=-\cos\alpha\hat{1}-\sin\alpha\hat{3}, \tag{16}$$

as can be seen from Fig. 1. Then

$$\frac{d\hat{1}}{dt}=\boldsymbol{\gamma}\times\hat{1}=-\dot{\phi}\sin\alpha\hat{2}, \tag{17}$$

$$\frac{d\hat{2}}{dt}=\boldsymbol{\gamma}\times\hat{2}=\dot{\phi}\sin\alpha\hat{1}-\dot{\phi}\cos\alpha\hat{3}, \tag{18}$$

and

$$\frac{d\hat{3}}{dt}=\boldsymbol{\gamma}\times\hat{3}=\dot{\phi}\cos\alpha\hat{2}. \tag{19}$$

It immediately follows that

$$\hat{1}\times\frac{d\hat{1}}{dt}=-\dot{\phi}\sin\alpha\hat{3}. \tag{20}$$

Comparing with Eq. (11) we see that $\omega_3=-\dot{\phi}\sin\alpha$ and hence from Eq. (14) we find

$$\boldsymbol{\omega}=(r'/a)\dot{\phi}\hat{1}-\dot{\phi}\sin\alpha\hat{3}. \tag{21}$$

As anticipated, the rolling constraint specifies how ω_1 and ω_3 are both related to the angular velocity $\dot{\phi}$ of the wheel about the $\hat{z}'$ axis.

For use in the equation of motion (7) we can now write

$$\mathbf{L}=\overleftrightarrow{\mathbf{I}}\cdot\boldsymbol{\omega}=kma^2[2(r'/a)\dot{\phi}\hat{1}-\dot{\phi}\sin\alpha\hat{3}], \tag{22}$$

and hence,

$$\frac{1}{ma}\,\frac{d\mathbf{L}}{dt}=2kr'\ddot{\phi}\hat{1}-k\dot{\phi}^2\sin\alpha(2r'+a\cos\alpha)\hat{2}$$
$$-ka\ddot{\phi}\sin\alpha\hat{3}, \tag{23}$$

using Eqs. (17)–(19). Also, by differentiating Eq. (13) we find

$$\frac{d^2\mathbf{r}_{\rm cm}}{dt^2}=r'\dot{\phi}^2\sin\alpha\hat{1}+r'\ddot{\phi}\hat{2}-r'\dot{\phi}^2\cos\alpha\hat{3}, \tag{24}$$

so that

$$\hat{3}\times\frac{d^2\mathbf{r}_{\rm cm}}{dt^2}=-r'\ddot{\phi}\hat{1}+r'\dot{\phi}^2\sin\alpha\hat{2}. \tag{25}$$

Combining (7), (23), and (25) the equation of motion reads

$$g\hat{z}\times\hat{3}=\hat{3}\times\frac{d^2\mathbf{r}_{\rm cm}}{dt^2}-\frac{1}{ma}\,\frac{d\mathbf{L}}{dt}$$
$$=-(2k+1)r'\ddot{\phi}1$$
$$+[(2k+1)r'+ka\cos\alpha]\dot{\phi}^2\sin\alpha\hat{2}$$
$$+ka\ddot{\phi}\sin\alpha\hat{3}. \tag{26}$$

To evaluate $\hat{z}\times\hat{3}$ we first express $\hat{z}$ in terms of the triad $(\hat{r}',\hat{2},\hat{z}')$, and then transform to triad $(\hat{1},\hat{2},\hat{3})$. When the point of contact of the wheel (and hence the $\hat{r}'$ axis) has azimuth ϕ relative to the $\hat{x}'$ axis, the $\hat{z}$ axis has azimuth $-\phi$ relative to the $\hat{r}'$ axis. Hence,

$$\begin{aligned}\hat{z}&=\sin\beta\cos\phi\hat{r}'-\sin\beta\sin\phi\hat{2}+\cos\beta\hat{z}'\\&=-(\cos\alpha\cos\beta+\sin\alpha\sin\beta\cos\phi)\hat{1}\\&\quad-\sin\beta\sin\phi\hat{2}\\&\quad-(\sin\alpha\cos\beta-\cos\alpha\sin\beta\cos\phi)\hat{3}\end{aligned}\tag{27}$$

using Eq. (16) and

$$\hat{r}'=\hat{2}\times\hat{z}'=-\sin\alpha\hat{1}+\cos\alpha\hat{3}.\tag{28}$$

Thus

$$\begin{aligned}\hat{z}\times\hat{3}=&-\sin\beta\sin\phi\hat{1}\\&+(\cos\alpha\cos\beta+\sin\alpha\sin\beta\cos\phi)\hat{2}.\end{aligned}\tag{29}$$

The $\hat{1}$, $\hat{2}$, and $\hat{3}$ components of the equation of motion are now

$$(2k+1)r'\ddot{\phi}=g\sin\beta\sin\phi,\tag{30}$$

$$\begin{aligned}&[(2k+1)r'+ka\cos\alpha]\dot{\phi}^2\sin\alpha\\&\quad=g(\cos\alpha\cos\beta+\sin\alpha\sin\beta\cos\phi),\end{aligned}\tag{31}$$

and

$$ka\ddot{\phi}\sin\alpha=0.\tag{32}$$

The cone angle θ enters the equations of motion only through r'.

1. Vertical orbits

From Eq. (32) we learn that for circular orbits either $\sin\alpha=0$ or $\ddot{\phi}=0$. We first consider the simpler case that $\sin\alpha=0$, which implies that the plane of the wheel lies in the plane of the orbit. For a wheel inside the sphere with $\sin\alpha=0$, we must have $\alpha=0$ to satisfy the geometric constraint (3). Then Eq. (31) can only be satisfied if $\cos\beta=0$, i.e., $\beta=\pi/2$ and the plane of the orbit is vertical. The remaining equation of motion (30) now reads

$$(2k+1)r'\ddot{\phi}=g\sin\phi,\tag{33}$$

with $r'=r\sin\theta-a>0$, which integrates to

$$\frac{2k+1}{2}mr'^2(\dot{\phi}^2-\dot{\phi}_0^2)=mgr'(1-\cos\phi),\tag{34}$$

where $\dot{\phi}_0$ is the angular velocity at the top of the orbit at which $\phi=0$. Equation (34) expresses conservation of energy. The angular velocity $\boldsymbol{\omega}$ and the angular momentum $\mathbf{L}_{\rm cm}$ vary in magnitude but are always perpendicular to the plane of the orbit.

The requirement that the wheel stay in contact with the sphere is that the contact force $\mathbf{F}$ have the component $F_\perp$ that points to the center of the sphere. On combining Eqs. (5), (24), (28), and (33) we find

$$\mathbf{F}=\frac{2k}{2k+1}mg\sin\phi\hat{2}+m(g\cos\phi-r'\dot{\phi}^2)\hat{3}.\tag{35}$$

The contact force is in the plane of the orbit, so the resulting torque about the center of mass of the wheel changes the magnitude but not the direction of the angular momentum. On the vertical orbits, axis $\hat{2}$ is tangent to the sphere, and axis $\hat{3}$ makes angle $\pi/2-\theta$ to the radius from the center of the sphere to the point of contact. Hence,

$$F_\perp=-F_3\sin\theta\tag{36}$$

is positive and the orbit is physical so long as the angular velocity $\dot{\phi}_0$ at the peak of the orbit obeys

$$\dot{\phi}_0^2>\frac{g}{r'},\tag{37}$$

as readily deduced from elementary considerations as well.

The required coefficient μ of static friction is given by $\mu=F_\parallel/F_\perp$, where

$$F_\parallel=\sqrt{F_3^2\cos\theta^2+F_2^2}\tag{38}$$

is the component of the contact force parallel to the surface of the sphere. We see that

$$\mu=\cot\theta\sqrt{1+(F_2/F_3\cos\theta)^2},\tag{39}$$

which must be greater than $\cot\theta$, but only much greater if the wheel nearly loses contact at the top of the orbit. Hence, orbits with $\pi/4\lesssim\theta\leq\pi/2$ are consistent with the friction of typical rubber wheels, namely, $\mu\lesssim1$.

Because a wheel experiences friction at the point of contact, vertical orbits are possible with $\theta<\pi/2$. This is in contrast to the case of a particle sliding freely on the inside of a sphere for which the only vertical orbits are great circles $(\theta=\pi/2)$. The only restriction in the present case is that the wheel fits inside the sphere, i.e., $r\sin\theta>a$, and that the minimum angular velocity satisfy Eq. (37).

2. Horizontal orbits

The second class of orbits is defined by $\ddot{\phi}=0$, so that the angular velocity is constant, say $\dot{\phi}=\Omega$. From Eq. (30) we see that $\sin\beta=0$ and hence $\beta=0$ for these orbits, which implies that they are horizontal. Then Eq. (31) gives the relation between the required angular velocity Ω and the geometrical parameters of the orbit:

$$\begin{aligned}\Omega^2&=\frac{g\cot\alpha}{(2k+1)r'+ka\cos\alpha}\\&=\frac{g\cot\alpha}{(2k+1)r\sin\theta-(k+1)a\cos\alpha},\end{aligned}\tag{40}$$

recalling Eq. (4). Compare example 3, Sec. 244 of Routh[5] or Sec. 407 of Milne.[2] There are no steady horizontal orbits for which $\alpha=0$, i.e., for which the wheel lies in the plane of the orbit. For such an orbit the angular momentum would be constant, but the torque on the wheel would be nonzero in contradiction.

In the following we will find that horizontal orbits are possible only for $0<\alpha<\pi/2$.

First, the requirement that $\Omega^2>0$ for real orbits puts various restrictions on the parameters of the problem. We examine these for the four quadrants of angle α.

1. $0<\alpha<\pi/2$. Then $\cot\alpha>0$, so we must have

$$r' > -\frac{ka\cos\alpha}{2k+1}. \tag{41}$$

This is satisfied by all $r'>0$ and some $r'<0$. However, for the wheel to fit inside the sphere with $0<\alpha<\pi/2$, we can have $r'<0$ only for $\theta>\pi/2$ according to Eqs. (3) and (4).

2. $\pi/2<\alpha<\pi$. Then $\cos\alpha<0$ and $\cot\alpha<0$, so the numerator of (40) is negative and the denominator is positive. Hence, Ω is imaginary and there are no steady orbits in this quadrant.

3. $-\pi<\alpha<-\pi/2$. Then $\cos\alpha<0$ but $\cot\alpha>0$, so $\Omega^2>0$ and $r'>0$ and Eq. (40) imposes no to restriction. For the wheel to fit inside the sphere with α in this quadrant we must have $\theta<\pi/2$.

4. $-\pi/2<\alpha<0$. Then $\cot\alpha<0$, so we must have

$$r' < -\frac{ka\cos\alpha}{2k+1} < 0. \tag{42}$$

For the wheel to be inside the sphere with $r'<0$ and α in this quadrant we must have $\theta<\pi/2$.

To obtain further restrictions on the parameters we examine under what conditions the wheel remains in contact with the sphere. The contact force $\mathbf{F}$ is deduced from Eqs. (5), (24), and (28) to be

$$\begin{aligned}\mathbf{F}/m &= (-g\cos\alpha + r'\Omega^2\sin\alpha)\hat{1}\\ &\quad -(g\sin\alpha + r'\Omega^2\cos\alpha)\hat{3}.\end{aligned} \tag{43}$$

It is more useful to express $\mathbf{F}$ in components along the $\hat{r}$ and $\hat{\theta}$ axes, where $\hat{r}$ points away from the center of the sphere and $\hat{\theta}$ points toward increasing θ. The two sets of axes are related by a rotation about axis $\hat{2}$:

$$\begin{aligned}\hat{1} &= -\cos(\theta-\alpha)\hat{r} + \sin(\theta-\alpha)\hat{\theta},\\ \hat{3} &= \sin(\theta-\alpha)\hat{r} + \cos(\theta-\alpha)\hat{\theta},\end{aligned} \tag{44}$$

so that

$$\begin{aligned}\mathbf{F}/m &= -(r'\Omega^2\sin\theta - g\cos\theta)\hat{r}\\ &\quad -(r'\Omega^2\cos\theta + g\sin\theta)\hat{\theta} = -r'\Omega^2\hat{r}' + g\hat{z}.\end{aligned} \tag{45}$$

The second form of Eq. (45) follows directly from elementary considerations. The inward component of the contact force, $F_\perp = -F_r$, is positive and the orbits are physical provided

$$r'\Omega^2 > g\cot\theta. \tag{46}$$

There can be no orbits with $r'<0$ and $\theta<\pi/2$, which rules out orbits in quadrant 4 of α, i.e., for $-\pi/2>\alpha<0$.

Using Eq. (40) for Ω^2 in Eq. (46) we deduce that contact is maintained for orbits with $r'>0$ only if

$$\cot\alpha > [2k+1+k(a/r')\cos\alpha]\cot\theta. \tag{47}$$

For $r'<0$ the sign of the inequality is reversed.

In the third quadrant of α we have $\cos\alpha<0$, so inequality (47) can be rewritten with the aid of (4) as

$$\cot\alpha > \left(1+2k-\frac{k}{1+r\sin\theta/a|\cos\alpha|}\right)\cot\theta > \cot\theta. \tag{48}$$

However, in this quadrant inequality (3) tells us

$$\cot\alpha < \cot[\theta+\sin^{-1}(a/r)] < \cot\theta. \tag{49}$$

Hence, there can be no steady orbits with $-\pi<\alpha<-\pi/2$.

Thus steady horizontal orbits are possible only for $0<\alpha<\pi/2$. Furthermore, since the factor in brackets of inequality (47) is roughly 2 for a wheel, this kinematic constraint is somewhat stronger than the purely geometric relation (3). However, a large class of orbits remains with $\theta<\pi/2$ as well as $\theta>\pi/2$.

The coefficient of friction μ at the point of contact must be at least $F_\parallel/F_\perp$, where $F_\parallel = |F_\theta|$ from Eq. (45). (For $\theta>\pi/2$ and α near zero the tangential friction F_θ can sometimes point in the $+\theta$ direction.) Hence, we need

$$\mu \geq \frac{|r'\Omega^2\cos\theta + g\sin\theta|}{r'\Omega^2\sin\theta - g\cos\theta}. \tag{50}$$

The acceleration of the center of mass of the wheel is $r'\Omega^2$, so according to Eq. (40) the corresponding number of g's is

$$\frac{\cot\alpha}{2k+1+k(a/r')\cos\alpha}. \tag{51}$$

Table I. Parameters for horizontal circular orbits of a wheel of radius 0.3 m rolling inside a sphere of radius 3.0 m. The wheel has coefficient $k=1/2$ pertaining to its moment of inertia. The polar angle of the orbit is θ so orbits above the equator of the sphere have $\theta<90°$. The plane of the wheel makes angle α to the horizontal. The minimum coefficient of friction required to support the motion is μ. The magnitude of the horizontal acceleration of the center of mass is reported as the number of g's.

θ (deg)	α (deg)	μ	v_{cm} (m/s)	No. of g's
15	5	16.1	4.8	48
30	5	2.82	8.0	53
45	10	2.15	7.0	27
60	10	1.19	7.9	27
60	25	3.45	4.9	10
75	15	0.96	6.8	18
75	30	2.13	4.7	8
90	25	0.96	5.3	10
90	45	2.04	3.7	5
135	60	0.56	2.3	3

Table I lists parameters of several horizontal orbits for a sphere of size as might be found in a motorcycle circus. The coefficient of friction of rubber tires is of order one, so orbits more than a few degrees above the equator involve very strong accelerations. The head of the motorcycle rider is closer to the vertical axis of the sphere than is the center of the wheel, so the number of g's experienced by the rider is somewhat less than that given in the table.

Figure 5 illustrates the allowed values of the tilt angle α as a function of the angle θ of the plane of the orbit, for $a/r=0.1$ as in Table I.

From Eq. (40) we see that $\alpha=\pi/2$, $\Omega=0$ is a candidate ''orbit'' in the lower hemisphere. On such an ''orbit'' the wheel is standing vertically at rest, and is not stable against falling over. We infer that stability will only occur for Ω greater than some minimum value not revealed by the analysis thus far.

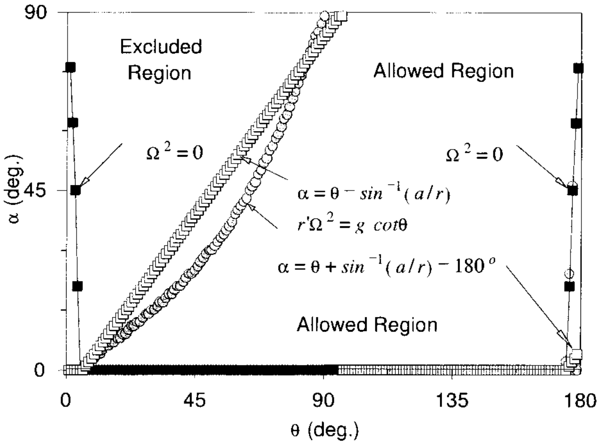

Fig. 5. The allowed values of the tilt angle α as a function of the angle θ of horizontal orbits for $a/r=0.1$. The allowed region is bounded by three curves, derived from expressions (3), (40), and (46).

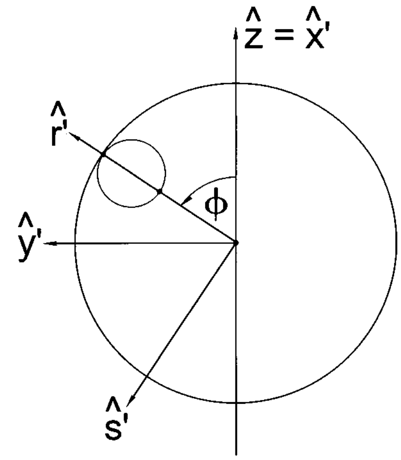

Fig. 6. For vertical orbits the x' axis is identical to the z axis. The axis $\hat{s}'=\hat{z}'\times\hat{r}'$ is in the direction of the unperturbed $\hat{2}$ axis.

C. Stability analysis

A completely general analysis of the stability of the steady circular orbits found above appears to be very difficult. We give a fairly general analysis for vertical orbits, but for horizontal orbits we obtain results only for orbits with $\theta=\pi/2$, i.e., orbits about the equator of the sphere and for orbits of ''small'' wheels.

We follow the approach of Sec. 405 of Milne,[2] where it was shown how the steady motion of a disk rolling in a straight line on a horizontal plane is stable if the angular velocity is great enough. It was also shown that the small oscillatory departures from steady motion lead to an oscillatory path of the point of contact of the wheel with the plane. Hence, in the present case we must consider perturbations that carry the wheel away from the plane of the steady orbit.

The difficulty is that there are, in general, four degrees of freedom for departures from steady motion: The axis of the wheel can be perturbed in two directions, the angular velocity $\dot{\phi}$ can be perturbed, as well as the angle θ to the point of contact. However, the procedure to eliminate the unknown force of contact from the six equations of motion of a rigid body leaves only three equations of motion. We will obtain solutions to the perturbed equations of motions only in special cases where there are, in effect, just two or three degrees of freedom. A more general analysis might be possible using the contact force found in steady motion as a first approximation to the contact force in perturbed motion, but we do not pursue this here.

A wheel rolling with a steady circular orbit on a plane can suffer only three types of perturbations and the results of an analysis are reported in example 3, Sec. 244 of Routh.[5] For a sphere rolling within a fixed sphere, the direction of what we call axis 3 always points to the center of the fixed sphere so there are only two perturbations to consider and the solution is relatively straightforward, as reviewed in Sec. IV below. The stability of horizontal orbits of rolling spheres lends confidence that stable orbits also exist for wheels.

1. Vertical orbits

We define the (x',y',z') coordinate system to have the x' axis vertical: $\hat{x}'=\hat{z}$. In steady motion we have

$$\alpha=0,\quad \hat{1}=-\hat{z}',\quad \hat{3}=\hat{r}'=\hat{x}'\cos\phi+\hat{y}'\sin\phi, \tag{52}$$

where ϕ is the azimuth of the center of the wheel from the $\hat{x}'$ axis. Thus $\phi=0$ at the top of the orbit. To discuss departures from steady motion in which the $\hat{1}$ axis is no longer parallel to the $\hat{z}'$ axis, it is useful to have a unit triad $(\hat{r}',\hat{s}',\hat{z}')$ defined by

$$\hat{r}'=\hat{x}'\cos\phi+\hat{y}\sin\phi,$$
$$\hat{s}'=\hat{z}'\times\hat{r}'=-\hat{x}'\sin\phi+\hat{y}'\cos\phi, \tag{53}$$

with ϕ defined as before. See Fig. 6. The surface of the sphere at the point of contact is parallel to the $s'-z'$ plane. Axes $\hat{r}'$ and $\hat{z}'$ rotate about the z' axis with angular velocity $\dot{\phi}$, so that

$$\frac{d\hat{r}'}{dt}=\dot{\phi}\hat{s}',\quad \frac{d\hat{s}'}{dt}=-\dot{\phi}\hat{r}'. \tag{54}$$

The perturbed $\hat{1}$ axis can then be written

$$\hat{1}=\epsilon_r\hat{r}'+\epsilon_s\hat{s}'-\hat{z}', \tag{55}$$

with $|\epsilon_r|,|\epsilon_s|\ll 1$, where throughout the stability analysis we ignore second-order terms. Writing

$$\hat{3}=\hat{r}'+\delta_s\hat{s}'+\delta_z\hat{z}', \tag{56}$$

with $|\delta_s|,|\delta_z|\ll 1$, the condition $\hat{1}\cdot\hat{3}=0$ requires that $\delta_z=\epsilon_r$. Then to first order,

$$\hat{2}=\hat{3}\times\hat{1}=-\delta_s\hat{r}'+\hat{s}'+\epsilon_s\hat{z}'. \tag{57}$$

We expect that vector $\hat{2}$ will remain parallel to the surface of the sphere even for large departure from steady motion, so $\hat{2}$ must remain in the $s'-z'$ plane. Hence, $\delta_s=0$ and

$$\hat{3}=\hat{r}'+\epsilon_r\hat{z}'. \tag{58}$$

Also, we can identify α as the tilt angle of the $\hat{3}$ axis to the $r'-s'$ plane, so that

$$\alpha=\epsilon_r. \tag{59}$$

The analysis proceeds along the lines of Sec. II B except that now we express all vectors in terms of the triad $(\hat{r}',\hat{s}',\hat{z}')$. To the first approximation the angular velocity of the wheel about the $\hat{1}$ axis is still given by $\omega_1=(r'/a)\dot{\phi}$. From Eqs. (54) and (55) we find

$$\frac{d\hat{1}}{dt}=(\dot{\epsilon}_r-\epsilon_s\dot{\phi})\hat{r}'+(\epsilon_r\dot{\phi}+\dot{\epsilon}_s)\hat{s}', \tag{60}$$

$$\hat{1}\times\frac{d\hat{1}}{dt}=(\epsilon_r\dot{\phi}+\dot{\epsilon}_s)\hat{r}'-(\dot{\epsilon}_r-\epsilon_s\dot{\phi})\hat{s}', \tag{61}$$

so that Eq. (11) yields

$$\boldsymbol{\omega}=\omega_1\hat{1}+\hat{1}\times\frac{d\hat{1}}{dt}=[(1+r'/a)\epsilon_r\dot{\phi}+\dot{\epsilon}_s]\hat{r}'$$
$$-[\dot{\epsilon}_r-(1+r'/a)\epsilon_s\dot{\phi}]\hat{s}'-(r'/a)\dot{\phi}\hat{z}'. \tag{62}$$

Then Eq. (12) tells us

$$\frac{\mathbf{L}}{ma}=2ka\omega_1\hat{1}+ka\hat{1}\times\frac{d\hat{1}}{dt}$$
$$=k[(2r'+a)\epsilon_r\dot{\phi}+a\dot{\epsilon}_s]\hat{r}'$$
$$-k[a\dot{\epsilon}_r-(2r'+a)\epsilon_s\dot{\phi}]\hat{s}'-2kr'\dot{\phi}\hat{z}', \tag{63}$$

so that to first order of smallness

$$\frac{1}{ma}\frac{d\mathbf{L}}{dt}$$
$$=k[2(r'+a)\dot{\epsilon}_r\dot{\phi}+(2r'+a)(\epsilon_r\ddot{\phi}-\epsilon_s\dot{\phi}^2)+a\ddot{\epsilon}_s]\hat{r}'$$
$$-k[a\ddot{\epsilon}_r-(2r'+a)(\epsilon_r\dot{\phi}^2-\epsilon_s\ddot{\phi})-2(r'+a)\dot{\epsilon}_s\dot{\phi}]\hat{s}'$$
$$-2k(r'\ddot{\phi}+\dot{r}'\dot{\phi})\hat{z}'. \tag{64}$$

In this we have noted from Eq. (4) that $\dot{r}'=r\dot{\theta}\sin\theta$ to first order and that $\dot{\theta}$ is small. Next,

$$\frac{d\mathbf{r}_{\rm cm}}{dt}=a\hat{3}\times\boldsymbol{\omega}\approx a\hat{r}'\times\boldsymbol{\omega}$$
$$=r'\dot{\phi}\hat{s}'-[a\dot{\epsilon}_r-(r'+a)\epsilon_s\dot{\phi}]\hat{z}'. \tag{65}$$

Then to first order,

$$\frac{d^2\mathbf{r}_{\rm cm}}{dt^2}=-r'\dot{\phi}^2\hat{r}'+(r'\ddot{\phi}+\dot{r}'\dot{\phi})\hat{s}'$$
$$-[a\ddot{\epsilon}_r-(r'+a)(\dot{\epsilon}_s\dot{\phi}+\epsilon_s\ddot{\phi})]\hat{z}', \tag{66}$$

so that

$$\hat{3}\times\frac{d^2\mathbf{r}_{\rm cm}}{dt^2}=(\hat{r}'+\epsilon_r\hat{z}')\times\frac{d^2\mathbf{r}_{\rm cm}}{dt^2}$$
$$=-r'\epsilon_r\ddot{\phi}\hat{r}'$$
$$+[a\ddot{\epsilon}_r-r'\epsilon_r\dot{\phi}^2-(r'+a)(\dot{\epsilon}_s\dot{\phi}+\epsilon_s\ddot{\phi})]\hat{s}'$$
$$+(r'\ddot{\phi}+\dot{r}'\dot{\phi})\hat{z}'. \tag{67}$$

Also,

$$\hat{3}\times\hat{z}=(\hat{r}'+\epsilon_r\hat{z}')\times(\cos\phi\hat{r}'-\sin\phi\hat{s}')$$
$$=\epsilon_r\sin\phi\hat{r}'+\epsilon_s\cos\phi\hat{s}'-\sin\phi\hat{z}'. \tag{68}$$

The r', s', and z' components of the equation of motion (7) are then

$$0=[(2k+1)r'+ka]\epsilon_r\ddot{\phi}+2k(r'+a)\dot{\epsilon}_r\dot{\phi}-g\epsilon_r\sin\phi$$
$$-k(2r'+a)\epsilon_s\dot{\phi}^2+ka\ddot{\epsilon}_s, \tag{69}$$

$$0=[(2k+1)r+ka]\epsilon_r\dot{\phi}^2-(k+1)a\ddot{\epsilon}_r-g\epsilon_r\cos\phi$$
$$+(2k+1)(r'+a)\dot{\epsilon}_s\dot{\phi}$$
$$+[(2k+1)r'+(k+1)a]\epsilon_s\ddot{\phi}, \tag{70}$$

$$0=(2k+1)(r'\ddot{\phi}+\dot{r}'\dot{\phi})-g\sin\phi. \tag{71}$$

If the perturbations ϵ_r, ϵ_s, and $\dot{r}'$ are set to zero, Eqs. (69) and (70) become trivial while Eq. (71) becomes the steady equation of motion (33).

The general difficulty with this analysis is that there are only three equations, (69)–(71), while there are four perturbations, ϵ_r, ϵ_s, $\ddot{\phi}$, and $\dot{\theta}$. The perturbation $\dot{\theta}$ appears only in Eq. (71) via $\dot{r}'$; Its effect on r' leads only to second-order terms in Eqs. (69) and (70). If we could neglect the terms in $\ddot{\phi}$ in Eqs. (69) and (70), then these two equations would describe only the perturbations ϵ_r and ϵ_s to first order and a solution could be completed.

Therefore, we restrict our attention to the top of the orbit, $\phi=0$, where Eq. (71) tells us that $\ddot{\phi}=0$ to leading order. The angular velocity $\dot{\phi}_0$ at this point is a minimum, so the gyroscopic stability of the wheel is the least here. Hence, if the orbit is stable at $\phi=0$ it will be stable at all ϕ.

The forms of Eqs. (69) and (70) for $\phi=0$ indicate that if ϵ_r and ϵ_s are oscillatory, then they are 90° out of phase. Therefore, we seek solutions

$$\epsilon_r=\epsilon_r\cos\omega t,\quad \epsilon_s=\epsilon_s\sin\omega t, \tag{72}$$

where ω now represents the oscillation frequency. The coupled equations of motion then yield the simultaneous linear equations

$$2k(r'+a)\dot{\phi}_0\omega\epsilon_r+[ka\omega^2+k(2r'+a)\dot{\phi}_0^2]\epsilon_s=0,$$
$$\{(k+1)a\omega^2+[(2k+1)r'+ka]\dot{\phi}_0^2-g\}\epsilon_r$$
$$+(2k+1)(r'+a)\dot{\phi}_0\omega\epsilon_s=0. \tag{73}$$

These equations are consistent only if the determinant of the coefficient matrix vanishes, which leads to the quadratic equation

$$A\omega^4-B\omega^2-C=0, \tag{74}$$

with solutions

$$\omega^2=\frac{B\pm\sqrt{B^2+4AC}}{2A}, \tag{75}$$

where

$$A=k(k+1)a^2, \tag{76}$$

$$B=kag+k[(2k+1)(2r'^2+a^2)+(4k+1)ar']\dot{\phi}_0^2, \tag{77}$$

and

$$C=k(2r'+a)([(2k+1)r'+ka]\dot{\phi}_0^2-g)\dot{\phi}_0^2. \tag{78}$$

Since A and B are positive, there are real, positive roots whenever B^2+4AC is positive, i.e., for $C>-B^2/4A$. In particular, this is satisfied for positive C, or equivalently for

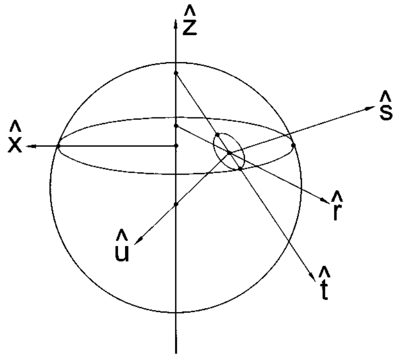

Fig. 7. For horizontal orbits of a wheel rolling inside a sphere, the (x,y,z) axes are identical with the (x',y',z') axes. The $\hat{r}-\hat{s}$ plane is horizontal. The axes $\hat{u}$, $\hat{s}$, and $\hat{t}$ are along the unperturbed directions of the $\hat{1}$, $\hat{2}$, and $\hat{3}$ axes, respectively. Axes $\hat{t}$ and $\hat{u}$ lie in the vertical plane $\hat{r}-\hat{z}$.

$$\dot{\phi}_0^2 > \frac{g}{(2k+1)r' + ka}. \tag{79}$$

However, this is less restrictive than the elementary result (37) that the wheel stay in contact with the sphere! All vertical orbits for which the wheel remains in contact with the sphere are stable against small perturbations.

The stability analysis yields the formal result that if $(\phi,\dot{\phi})=(0,0)$, then $\omega=\sqrt{g/(k+1)a}$. We recognize this as the frequency of oscillation of a simple pendulum formed by suspending the wheel from a point on its rim, the motion being perpendicular to the plane of the wheel.

2. Horizontal orbits

We expect the stability analysis of horizontal orbits to be nontrivial since we have identified steady orbits that are ''obviously'' unstable.

The spirit of the analysis has been set forth in the preceding sections. For horizontal orbits the (x',y',z') coordinate system can be taken as identical to the (x,y,z) system, so we drop the prime symbol in this section. We introduce a triad $(\hat{r},\hat{s},\hat{z})$, with $\hat{r}$ being the perpendicular unit vector from the z axis toward the center of the wheel. Then $\hat{s}$ points in the direction of the motion of the center of the wheel in case of steady motion.

It is also useful to introduce a unit triad that points along the $(\hat{1},\hat{2},\hat{3})$ axes for steady motion. The $\hat{s}$ axis already points along the $\hat{2}$ axis for steady motion, so we only need define $\hat{t}$ as being along the direction of $\hat{3}$ and $\hat{u}$ as being along the direction of $\hat{1}$ for steady motion, as shown in Fig. 7. Then $(\hat{s},\hat{t},\hat{u})$ form a right-handed unit triad. The vertical, $\hat{z}$, is then related by

$$\hat{z} = -\sin\alpha_0\hat{t} - \cos\alpha_0\hat{u}, \tag{80}$$

where α_0 is the angle of inclination of the wheel to the horizontal in steady motion. The triad $(\hat{s},\hat{t},\hat{u})$ rotates about the $\hat{z}$ axis with angular velocity $\dot{\phi}$, so that

$$\frac{d\hat{s}}{dt} = \dot{\phi}\hat{z}\times\hat{s} = -\dot{\phi}\cos\alpha_0\hat{t} + \dot{\phi}\sin\alpha_0\hat{u}, \tag{81}$$

$$\frac{d\hat{t}}{dt} = \dot{\phi}\cos\alpha_0\hat{s}, \tag{82}$$

and

$$\frac{d\hat{u}}{dt} = -\dot{\phi}\sin\alpha_0\hat{s}. \tag{83}$$

We now consider small departures from steady motion. The $\hat{1}$ axis deviates slightly from the $\hat{u}$ axis according to

$$\hat{1} = \epsilon_s\hat{s} + \epsilon_t\hat{t} + \hat{u}, \quad |\epsilon_s|,|\epsilon_t| \ll 1. \tag{84}$$

The $\hat{3}$ axis departs slightly from the t axis, but to the first approximation it remains in a vertical plane, i.e., the $t-u$ plane. Then we have

$$\hat{2} = \hat{s} - \epsilon_s\hat{1}, \quad \hat{3} = \hat{t} - \epsilon_t\hat{u}. \tag{85}$$

With the above definitions the signs of angles α and ϵ_t are opposite:

$$\Delta\alpha = -\epsilon_t, \quad \dot{\alpha} = -\dot{\epsilon}_t. \tag{86}$$

To first approximation the component ω_1 of the angular velocity of the wheel about its axis remains $\omega_1 = (r'/a)\dot{\phi}$. Then

$$\frac{d\hat{1}}{dt} = (-\dot{\phi}\sin\alpha_0 + \dot{\epsilon}_s - \epsilon_t\dot{\phi}\cos\alpha_0)\hat{s} - (\epsilon_s\dot{\phi}\cos\alpha_0 - \dot{\epsilon}_t)\hat{t} + \epsilon_s\dot{\phi}\sin\alpha_0\hat{u}, \tag{87}$$

$$\hat{1}\times\frac{d\hat{1}}{dt} = (\epsilon_s\dot{\phi}\cos\alpha_0 - \dot{\epsilon}_t)\hat{s} - (\dot{\phi}\sin\alpha_0 + \dot{\epsilon}_s - \epsilon_t\dot{\phi}\cos\alpha_0)\hat{t} + \epsilon_t\dot{\phi}\sin\alpha_0\hat{u}, \tag{88}$$

so that

$$\begin{aligned}\boldsymbol{\omega} &= \omega_1\hat{1} + \hat{1}\times\frac{d\hat{1}}{dt} \\ &= [(r'/a + \cos\alpha_0)\epsilon_s\dot{\phi} - \dot{\epsilon}_t]\hat{s} \\ &\quad - [\dot{\phi}\sin\alpha_0 - \dot{\epsilon}_s - (r'/a + \cos\alpha_0)\epsilon_t\dot{\phi}]\hat{t} \\ &\quad + (r'/a + \epsilon_t\sin\alpha_0)\dot{\phi}\hat{u},\end{aligned} \tag{89}$$

and

$$\begin{aligned}\frac{\mathbf{L}}{ma} &= 2ka\omega_1\hat{1} + ka\hat{1}\times\frac{d\hat{1}}{dt} \\ &= k[(2r' + a\cos\alpha_0)\epsilon_s\dot{\phi} - a\dot{\epsilon}_t]\hat{s} \\ &\quad - k[a\dot{\phi}\sin\alpha_0 - a\dot{\epsilon}_s - (2r' + a\cos\alpha_0)\epsilon_t\dot{\phi}]\hat{t} \\ &\quad + k(2r' + a\epsilon_t\sin\alpha_0)\dot{\phi}\hat{u}.\end{aligned} \tag{90}$$

Then, to the first approximation,

$$\frac{1}{ma}\frac{d\mathbf{L}}{dt}$$

$$=-k[(2r'+a\cos\alpha_0)\dot{\phi}^2\sin\alpha_0$$
$$-2(r'+a\cos\alpha_0)\dot{\epsilon}_s\dot{\phi}$$
$$-(2r'\cos\alpha_0+a\cos 2\alpha_0)\epsilon_t\dot{\phi}^2+a\ddot{\epsilon}_t]\hat{s}$$
$$-k[a\ddot{\phi}\sin\alpha_0+(2r'+a\cos\alpha_0)\epsilon_s\dot{\phi}^2\cos\alpha_0$$
$$-a\ddot{\epsilon}_s-2(r'+a\cos\alpha_0)\dot{\epsilon}_t\dot{\phi}]\hat{t}$$
$$+k[2r'\ddot{\phi}+2\dot{r}'\dot{\phi}+(2r'+a\cos\alpha_0)\epsilon_s\dot{\phi}^2\sin\alpha_0]\hat{u}. \quad (91)$$

Unlike the case of vertical orbits, for horizontal orbits the factor $\ddot{\phi}$ has no zeroth-order component and we neglect terms like $\epsilon\ddot{\phi}$.

Similarly,

$$\frac{d\mathbf{r}_{\rm cm}}{dt}=a\hat{3}\times\boldsymbol{\omega}$$
$$=a(\hat{t}-\epsilon_t\hat{u})\times\boldsymbol{\omega}$$
$$=r'\dot{\phi}\hat{s}-[(r'+a\cos\alpha_0)\epsilon_s\dot{\phi}-a\dot{\epsilon}_t]\hat{u}, \quad (92)$$

$$\frac{d^2\mathbf{r}_{\rm cm}}{dt^2}=[r'\ddot{\phi}+\dot{r}'\dot{\phi}+(r'+a\cos\alpha_0)\epsilon_s\dot{\phi}^2\sin\alpha_0$$
$$-a\dot{\epsilon}_t\dot{\phi}\sin\alpha_0]\hat{s}-r'\dot{\phi}^2\cos\alpha_0\hat{t}$$
$$+[r'\dot{\phi}^2\sin\alpha_0-(r'+a\cos\alpha_0)\dot{\epsilon}_s\dot{\phi}+a\ddot{\epsilon}_t]\hat{u}, \quad (93)$$

and

$$\hat{3}\times\frac{d^2\mathbf{r}_{\rm cm}}{dt^2}=(\hat{t}-\epsilon_t\hat{u})\times\frac{d^2\mathbf{r}_{\rm cm}}{dt^2}$$
$$=[r'\dot{\phi}^2\sin\alpha_0-(r'+a\cos\alpha_0)\dot{\epsilon}_s\dot{\phi}$$
$$-r'\epsilon_t\dot{\phi}^2\cos\alpha_0+a\ddot{\epsilon}_t]\hat{s}-[r'\ddot{\phi}+\dot{r}'\dot{\phi}$$
$$+(r'+a\cos\alpha)\epsilon_s\dot{\phi}^2\sin\alpha_0-a\dot{\epsilon}_t\dot{\phi}\sin\alpha_0]\hat{u}. \quad (94)$$

We also need

$$\hat{3}\times\hat{z}=(\hat{t}-\epsilon_t\hat{u})\times(-\sin\alpha_0\hat{t}-\cos\alpha_0\hat{u})$$
$$=-(\cos\alpha_0+\epsilon_t\sin\alpha_0)\hat{s}. \quad (95)$$

The s, t, and u components of the equation of motion (7) are

$$0=[(2k+1)r'+ka\cos\alpha_0]\dot{\phi}^2\sin\alpha_0-g\cos\alpha_0$$
$$-(2k+1)(r'+a\cos\alpha_0)\dot{\epsilon}_s\dot{\phi}-[(2k+1)r'\cos\alpha_0$$
$$+ka\cos 2\alpha_0]\epsilon_t\dot{\phi}^2-g\epsilon_t\sin\alpha_0+(k+1)a\ddot{\epsilon}_t, \quad (96)$$

$$0=ka\ddot{\phi}\sin\alpha_0+k(2r'+a\cos\alpha_0)\epsilon_s\dot{\phi}^2\sin\alpha_0-ka\ddot{\epsilon}_s$$
$$-2k(r'+a\cos\alpha_0)\dot{\epsilon}_t\dot{\phi}, \quad (97)$$

and

$$0=(2k+1)(r'\ddot{\phi}+\dot{r}'\dot{\phi})+[(2k+1)r'$$
$$+(k+1)a\cos\alpha_0]\epsilon_s\dot{\phi}^2\sin\alpha_0-a\dot{\epsilon}_t\dot{\phi}\sin\alpha_0. \quad (98)$$

The leading terms of these three equations are just Eqs. (30)–(32) for $\beta=0$. Therefore we can write $\dot{\phi}=\Omega+\dot{\delta}$, where Ω is the angular velocity of the steady horizontal orbit and δ is a small correction.

Although the derivative of r',

$$\dot{r}'=r\dot{\theta}\cos\theta_0+a\dot{\alpha}\sin\alpha_0=r\dot{\theta}\cos\theta_0-a\dot{\epsilon}_t\sin\alpha_0, \quad (99)$$

appears only in Eq. (98), in general the perturbation $\dot{\theta}$ is not decoupled from ϵ_s and ϵ_t, as was the case for vertical orbits. Thus far, we have found a way to proceed only in somewhat special cases in which the θ perturbation can be ignored, as described in Secs. II C 3 and II C 4.

3. Orbits near the equator

It appears possible to carry the analysis forward for the special case $\theta_0=\pi/2$, the orbit on the equator of the sphere. This case is, however, of interest.

Assuming $\theta_0=\pi/2$, the equations of motion (96)–(98) then provide three relations among the three perturbations δ, ϵ_s, and ϵ_t. For this we consider only the first-order terms, noting that $\dot{\phi}^2\approx\Omega^2+2\Omega\dot{\delta}$ and

$$r'=r\sin\theta-a\cos\alpha\approx r_0'+r\Delta\theta\cos\theta_0+a\Delta\alpha\sin\alpha_0$$
$$=r_0'-a\epsilon_t\sin\alpha_0, \quad (100)$$

where $r_0=r-a\cos\alpha_0$ for $\theta_0=\pi/2$, recalling Eq. (86). Also, from the form of Eqs. (96)–(98) we infer that if the perturbations are oscillatory, then δ and ϵ_s have the same phase, which is 90° from that of ϵ_t. Therefore, we seek solutions of the form

$$\delta=\delta\sin\omega t,\quad \epsilon_s=\epsilon_s\sin\omega t,\quad \epsilon_t=\epsilon_t\cos\omega t, \quad (101)$$

where ω is the frequency of oscillation. The first-order terms of the differential equations (96)–(98) then yield the algebraic relations

$$0=2\Omega\sin\alpha_0[(2k+1)r_0'+ka\cos\alpha_0]\omega\delta$$
$$-\Omega(2k+1)(r_0'+a\cos\alpha_0)\omega\epsilon_s$$
$$-\{\Omega^2[(2k+1)r_0'\cos\alpha_0+(k+\sin^2\alpha_0)a]$$
$$-g\sin\alpha_0+(k+1)a\omega^2\}\epsilon_t,$$
$$0=-ka\sin\alpha_0\omega^2\delta$$
$$+[k\Omega^2\cos\alpha_0(2r_0'+a\cos\alpha_0)+ka\omega^2]\epsilon_s$$
$$+2k\Omega(r_0'+a\cos\alpha_0)\omega\epsilon_t, \quad (102)$$
$$0=-(2k+1)r_0'\omega^2\delta$$
$$+\Omega^2\sin\alpha_0[(2k+1)r_0'+(k+1)a\cos\alpha_0]\epsilon_s$$
$$+2(k+1)\Omega\sin\alpha_0 a\omega\epsilon_t.$$

These equations have the form

$$
\begin{aligned}
&A_{11}\omega\delta \qquad\qquad +A_{12}\omega\epsilon_s \quad +(A_{13}+B_{13}\omega^2)\epsilon_t=0,\\
&A_{21}\omega^2\delta \quad +(A_{22}+B_{22}\omega^2)\epsilon_s \qquad\qquad +A_{23}\omega\epsilon_t=0,\\
&A_{31}\omega^2\delta \qquad\qquad +A_{32}\epsilon_s \qquad\qquad +A_{33}\omega\epsilon_t=0.
\end{aligned} \tag{103}
$$

To have consistency the determinant of the coefficient matrix must vanish, which leads quickly to the quadratic equation

$$A\omega^4-B\omega^2-C=0, \tag{104}$$

where

$$A=B_{13}B_{22}A_{31}, \tag{105}$$

$$
\begin{aligned}
B=&A_{11}B_{22}A_{33}+A_{12}A_{23}A_{31}+B_{13}A_{21}A_{32}-A_{13}B_{22}A_{31}\\
&-B_{13}A_{22}A_{31}-A_{12}A_{21}A_{33},
\end{aligned} \tag{106}
$$

and

$$C=A_{11}A_{22}A_{33}+A_{13}A_{21}A_{32}-A_{13}A_{22}A_{31}-A_{11}A_{23}A_{32}. \tag{107}$$

From numerical evaluation it appears that A, B, and C are all positive for angular velocities Ω that obey Eq. (40). That is, all steady orbits at the equator of the sphere are stable. There is both a fast and slow oscillation about steady motion for these orbits, an effect familiar from nutations of a symmetric top.

4. Small wheel inside a large sphere

The analysis can also be carried further in the approximation that the radius a of the wheel is much less than the radius r of the fixed sphere. In this case the perturbation in angle θ of the orbit is of higher order than the perturbations in azimuth ϕ and in the angles ϵ_s and ϵ_t related to the axes of the wheel. A solution describing the three first-order perturbations can then be obtained.

For the greatest simplification we also require that

$$a \ll r_0' \approx r \sin\theta_0. \tag{108}$$

Thus we restrict our attention to orbits significantly different from the special cases of motion near the poles of the fixed sphere.

In the present approximation the first-order terms of the perturbed equations of motion (96)–(98) are

$$2\Omega\dot{\delta}\sin\alpha_0=\Omega\dot{\epsilon}_s+\left(\Omega^2\cos\alpha_0+\frac{g\sin\alpha_0}{(2k+1)r_0'}\right)\epsilon_t, \tag{109}$$

$$\epsilon_s=\frac{\dot{\epsilon}_t}{\Omega\sin\alpha_0}, \tag{110}$$

and

$$\ddot{\delta}=-\epsilon_s\Omega^2\sin\alpha_0. \tag{111}$$

Inserting (110) into (111) we can integrate the latter to find

$$\dot{\delta}=-\Omega\epsilon_t. \tag{112}$$

Using this and the derivative of (110) in (109) we find that ϵ_t obeys

$$\ddot{\epsilon}_t+\left[\Omega^2\sin\alpha_0(\cos\alpha_0+2\sin\alpha_0)+\frac{g\sin^2\alpha_0}{(2k+1)r_0'}\right]\epsilon_t=0. \tag{113}$$

Then the frequency ω of the perturbations is given by

$$
\begin{aligned}
\omega^2&=\Omega^2\sin\alpha_0(\cos\alpha_0+2\sin\alpha_0)+\frac{g\sin^2\alpha_0}{(2k+1)r_0'}\\
&=\Omega^2\tan\alpha_0(1+\sin 2\alpha_0),
\end{aligned} \tag{114}
$$

using Eqs. (40) and (108).

Thus all orbits for small wheels are stable if condition (108) holds. We conjecture that orbits for large wheels are also stable if (108) is satisfied.

For steady orbits that lie very near the poles, i.e., those that have $r_0' \lesssim a$, we conjecture that the motion is stable only for Ω greater than some minimum value. For a wheel spinning about its axis on a horizontal plane the stability condition is

$$\Omega^2>\frac{g}{(2k+1)a}. \tag{115}$$

See, for example, Sec. 55 of Deimel.[4] However, we have been unable to deduce the generalization of this constraint to include the dependence on r and θ_0 for small $r\sin\theta_0$.

III. WHEEL ROLLING OUTSIDE A FIXED SPHERE

Equations (1)–(32) hold for a wheel rolling outside a sphere as well as inside when the geometric relation (3) is rewritten as

$$\theta<\alpha<\pi+\theta. \tag{116}$$

We expect no vertical orbits, as the wheel will lose contact with the sphere at some point. To verify this, note that the condition $\sin\alpha=0$ [from Eq. (32)] implies that $\alpha=\pi$ when the wheel is outside the sphere. Then Eqs. (34)–(36) indicate, for example, that if the wheel starts from rest at the top of the sphere, it loses contact with the sphere when

$$\cos\phi=\frac{2}{3+2k}. \tag{117}$$

The result for a particle sliding on a sphere $(k=0)$ is well known.

For horizontal orbits, Eqs. (40)–(45) are still valid, but the condition that friction have an outward component is now

$$r'\Omega^2<g\cot\theta, \tag{118}$$

and hence,

$$\cot\alpha<(2k+1+k(a/r')\cos\alpha)\cot\theta. \tag{119}$$

Equation (40) can be satisfied for $\alpha<\pi/2$ so long as the radius of the wheel is small enough that $(2k+1)r'+ka\cos\alpha$ is positive. We must have $\theta<\pi/2$ to have $\alpha<\pi/2$ since $\alpha>\theta$, so horizontal orbits exist on the upper hemisphere. A particular solution is $\alpha=\pi/2$ for which $\Omega=0$; this is clearly unstable.

There is a class of orbits with $\theta<\pi/2$ and α very near $\pi+\theta$ that satisfy both Eqs. (40) and (119). These also appear to be unstable.

The stability analysis of the preceding section holds formally for wheels outside spheres, but the restriction there to the case of $\theta=90°$ provides no insight into the present case.

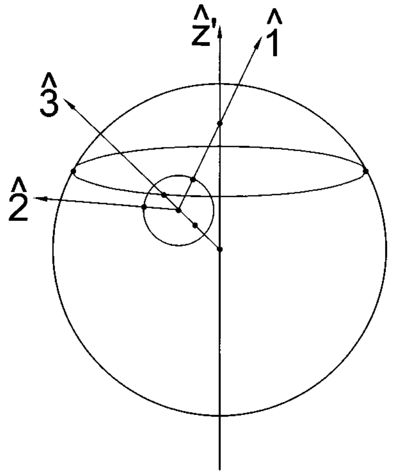

Fig. 8. Geometry illustrating the case of a sphere rolling without slipping on a circular orbit perpendicular to the $\hat{z}'$ axis inside a fixed sphere. The $\hat{3}$ axis is along the line of centers of the two spheres and passes through the point of contact. The $\hat{2}$ axis lies in the plane of the orbit along the direction of motion of the center of the rolling sphere and axis $\hat{1}=\hat{2}\times\hat{3}$ is in the $\hat{3}$–$\hat{z}'$ plane.

IV. SPHERE ROLLING INSIDE A FIXED SPHERE

The case of a sphere rolling on horizontal orbits inside a fixed sphere has been treated by Milne.[2] For completeness we give an analysis for orbits of arbitrary inclination to compare and contrast with the case of a wheel.

Again, the axis normal to the orbit is called $\hat{z}'$, which makes angle β to the vertical $\hat{z}$. The polar angle of the orbit about $\hat{z}'$ is θ and ϕ is the azimuth of the point of contact between the two spheres. The radius of the fixed sphere is r.

The diameter of the rolling sphere that passes through the point of contact must always be normal to the fixed sphere. That is, the ''bank'' angle of the rolling sphere is always $\theta - \pi/2$ with respect to the plane of the orbit.

The rolling sphere has radius a, mass m, and moment of inertia kma^2 about any diameter. The angular momentum is, of course,

$$\mathbf{L}_{\rm cm}=kma^2\boldsymbol{\omega}, \tag{120}$$

where $\boldsymbol{\omega}$ is the angular velocity of the rolling sphere.

We again introduce a right-handed triad of unit vectors $(\hat{1},\hat{2},\hat{3})$ centered on the rolling sphere. For consistency with the notation used for the wheel, axis $\hat{3}$ is directed toward the point of contact, axis $\hat{2}$ is parallel to the plane of the orbit, and axis $\hat{1}$ is in the $\hat{3}$–$\hat{z}'$ plane, as shown in Fig. 8. In general, none of these vectors are body vectors for the rolling sphere. The center of mass of the rolling sphere lies on the line joining the center of the fixed sphere to the point of contact, and so

$$\mathbf{r}_{\rm cm}=(r-a)\hat{3}\equiv r'\hat{3}. \tag{121}$$

Equations (5)–(10), which govern the motion and describe the rolling constraint, hold for the sphere as well as the wheel. Using Eqs. (120) and (121) we can write Eq. (7) as

$$ka\,\frac{d\boldsymbol{\omega}}{dt}=g\hat{3}\times\hat{z}+r'\hat{3}\times\frac{d^2\hat{3}}{dt^2}. \tag{122}$$

We seek an additional expression for the angular velocity $\boldsymbol{\omega}$ of the rolling sphere, but we cannot use Eq. (11) since we have not identified a body axis in the sphere. However, with Eq. (121) the rolling constraint (10) can be written

$$\boldsymbol{\omega}=-\frac{r'}{a}\,\hat{3}\times\frac{d\hat{3}}{dt}+\omega_3\hat{3}. \tag{123}$$

We can now see that $\omega_3=\boldsymbol{\omega}\cdot\hat{3}$ is a constant by noting that $\hat{3}\cdot d\boldsymbol{\omega}/dt=0$ from Eq. (122), and also $\boldsymbol{\omega}\cdot d\hat{3}/dt=0$ from Eq. (123). The freedom to chose the constant angular velocity ω_3 for a rolling sphere permits stable orbits above the equator of the fixed sphere, just as the freedom to adjust the bank angle α allows such orbits for a wheel.

Taking the derivative of Eq. (123) we find

$$\frac{d\boldsymbol{\omega}}{dt}=-\frac{r'}{a}\,\hat{3}\times\frac{d^2\hat{3}}{dt^2}+\omega_3\,\frac{d\hat{3}}{dt}, \tag{124}$$

so the equation of motion (122) can be written

$$(k+1)r'\hat{3}\times\frac{d^2\hat{3}}{dt^2}-ka\omega_3\,\frac{d\hat{3}}{dt}=g\hat{z}\times\hat{3}. \tag{125}$$

Milne notes that this equation is identical to that for a symmetric top with one point fixed,[2] and so the usual extensive analysis of nutations about the stable orbits follows if desired.

We again restrict ourselves to circular orbits, for which the angular velocity of the center of mass, and of $\hat{1}$, $\hat{2}$, and $\hat{3}$ is $\dot{\phi}\hat{z}'$ where the z' axis is fixed. Then with

$$\hat{z}'=-\sin\,\theta\hat{1}+\cos\,\theta\hat{3}, \tag{126}$$

we have

$$\frac{d\hat{3}}{dt}=\dot{\phi}\hat{z}'\times\hat{3}=\dot{\phi}\,\sin\,\theta\hat{2}, \tag{127}$$

$$\frac{d^2\hat{3}}{dt^2}=\dot{\phi}^2\,\sin\,\theta\hat{z}'\times\hat{2}+\ddot{\phi}\,\sin\,\theta\hat{2}$$
$$=-\dot{\phi}^2\,\sin\,\theta\,\cos\,\theta\hat{1}+\ddot{\phi}\,\sin\,\theta\hat{2}+\dot{\phi}^2\,\sin^2\,\theta\hat{3}, \tag{128}$$

and hence

$$\hat{3}\times\frac{d^2\hat{3}}{dt^2}=-\ddot{\phi}\,\sin\,\theta\hat{1}-\dot{\phi}^2\,\sin\,\theta\,\cos\,\theta\hat{2}. \tag{129}$$

With these, the equation of motion (125) reads

$$-g\hat{z}\times\hat{3}=(k+1)r'\ddot{\phi}\,\sin\,\theta\hat{1}$$
$$+[(k+1)r'\dot{\phi}^2\,\cos\,\theta+ka\omega_3\dot{\phi}]\sin\,\theta\hat{2}. \tag{130}$$

We can use Eq. (29) for $\hat{z}\times\hat{3}$ if we substitute $\alpha=\theta-\pi/2$ for the rolling sphere:

$$\hat{z}\times\hat{3}=-\sin\,\beta\,\sin\,\phi\hat{1}$$
$$+(\sin\,\theta\,\cos\,\beta-\cos\,\theta\,\sin\,\beta\,\cos\,\phi)\hat{2}. \tag{131}$$

The components of the equation of motion are then

$$(k+1)r'\ddot{\phi}\,\sin\,\theta=\sin\,\beta\,\sin\,\phi, \tag{132}$$

$$[(k+1)r'\dot{\phi}^2\,\cos\,\theta+ka\omega_3\dot{\phi}]\sin\,\theta$$
$$=g\,\cos\,\theta\,\sin\,\beta\,\cos\,\phi-g\,\sin\,\theta\,\cos\,\beta. \tag{133}$$

The two equations of motion are not consistent in general. To see this, take the derivative of Eq. (133) and substitute $\ddot{\phi}$ from Eq. (132):

$$ka\omega_3 \sin\beta \sin\phi = -3(k+1)r'\dot{\phi}\cos\theta \sin\beta \sin\phi. \tag{134}$$

While this is certainly true for $\beta=0$ (horizontal orbits), for nonzero β we must have $\dot{\phi}\cos\theta$ constant since ω_3 is constant. Equation (134) is satisfied for $\theta=\pi/2$ (great circles), but for arbitrary θ we would need $\dot{\phi}$ constant, which is inconsistent with Eq. (132). Further, on a great circle Eq. (133) becomes $ka\omega_3\dot{\phi}=-g\cos\beta$. This is inconsistent with Eq. (132) unless $\beta=\pi/2$ (vertical great circles) and $\omega_3=0$.

In summary, the only possible closed orbits for a sphere rolling within a fixed sphere are horizontal circles and vertical great circles.

We remark further only on the horizontal orbits. For these, $\dot{\phi}\equiv\Omega$ is constant according to Eq. (132). Equation (133) then yields a quadratic equation for Ω:

$$(k+1)r'\Omega^2\cos\theta + ka\omega_3\Omega + g = 0, \tag{135}$$

so that there are orbits with real values of Ω provided

$$(ka\omega_3)^2 \geqslant 4(k+1)gr'\cos\theta. \tag{136}$$

This is satisfied for orbits below the equator ($\theta>\pi/2$) for any value of the ''spin'' ω_3 of the sphere (including zero), but places a lower limit on $|\omega_3|$ for orbits above the equator. For the orbit on the equator we must have $\Omega=-g/(ka\omega_3)$, so a nonzero ω_3 is required here as well.

The contact force $\mathbf{F}$ is given by

$$\mathbf{F}/m = (g + r'\Omega^2\cos\theta)\sin\theta\hat{1} - (r'\Omega^2\sin^2\theta - g\cos\theta)\hat{3}, \tag{137}$$

using Eqs. (5) and (134). For the rolling sphere to remain in contact with the fixed sphere, there must be a positive component of $\mathbf{F}$ pointing toward the center of the fixed sphere. Since axis $\hat{3}$ is radial outward from the fixed sphere, we require that F_3 be negative and hence,

$$r'\Omega^2\sin^2\theta > g\cos\theta. \tag{138}$$

This is always satisfied for orbits below the equator. For orbits well above the equator this requires a larger value of $|\omega_3|$ than does Eq. (136). To see this, suppose ω_3 is exactly at the minimum value allowed by Eq. (136), which implies that $\Omega=-ka\omega_3/(2(k+1)r'\cos\theta)$. Then Eq. (138) requires that $\tan^2\theta>k+1$. So for $k=2/5$ and at angles $\theta<50°$, larger values of $|\omega_3|$ are needed to satisfy Eq. (136) than to satisfy Eq. (136). However, there are horizontal orbits at any $\theta>0$ for $|\omega_3|$ large enough.

V. SPHERE ROLLING OUTSIDE A FIXED SPHERE

This case has also been treated by Milne.[2] A popular example is spinning a basketball on one's fingertip.

Equations (135) and (136) hold with the substitution that $r'=r+a$. The condition on the contact force becomes

$$r'\Omega^2\sin^2\theta < g\cos\theta, \tag{139}$$

which can only be satisfied for $\theta<\pi/2$. While Eq. (136) requires a large spin $|\omega_3|$, if it is too large Eq. (139) can no longer be satisfied in view of relation (135). For any case in which the orbit exists a perturbation analysis shows that the motion is stable against small nutations.[2]

[1]M. A. Abramowicz and E. Szuszkiewicz, ''The Wall of Death,'' Am. J. Phys. **61**, 982–991 (1993).

[2]E. A. Milne, *Vectorial Mechanics* (Interscience, New York, 1948).

[3]H. Lamb, *Higher Mechanics* (Cambridge U.P., Cambridge, 1920).

[4]R. F. Deimel, *Mechanics of the Gyroscope* (Macmillian, London, 1929; reprinted by Dover, New York, 1950).

[5]E. J. Routh, *The Advanced Part of a Treatise on the Dynamics of a System of Rigid Bodies* (Macmillan, London, 1905, 6th ed.; reprinted by Dover, New York, 1955).

THE UNBELIEVABLE CONSEQUENCES OF IDENTITY

In conclusion, I would like to point out a paradox, the resolution of which eludes me. It is easy to apply the method given here to the case of a mixture of two different gases. In that case, each type of molecule has its own ''cells.'' From that follows the additivity of the entropies of the components of the mixture. Therefore, with regard to molecular energy, pressure, and statistical distribution, each component behaves as though it alone were present. At given temperature, a mixture with numbers n_1 and n_2 of molecules, wherein the molecules of type 1 and 2 differ from each other arbitrarily little (in particular, with respect to the molecular masses m_1 and m_2), produces therefore a different pressure and a different distribution over states than a homogeneous gas that has molecular number n_1+n_2, practically the same molecular mass, and the same volume. This appears, however, to be virtually impossible.

Albert Einstein, final paragraph of his first paper on the ''Quantum theory of a monatomic ideal gas,'' wherein he discovered the Bose-Einstein condensation. *Berliner Berichte*, 261–267 (1924) (Translation by Ralph Baierlein).