scale that best approximates the target just interval. Also, the numerators of the principal convergents yield the number of equal-tempered steps that correspond to the best approximation of the target just interval. Using other criteria and algorithms,[6,7] various authors have generated these same ETS, but a direct comparison of equal-tempered scales must be made to choose the one that best approximates the target just interval.

Although continued fraction principal convergents with large values of q more closely approximate the harmonious interval on which the scale is based, musically (especially for fixed string length or fretted instruments), we would confine our choices to relatively small values of q. With the advent of computer music and synthesizers this consideration is less important.

[1]Dwight E. Neuenschwander, "Question #4. Is there a physics application that is best analyzed in terms of continued fractions?," Am. J. Phys. **62** (10), 871 (1994).

[2]H. Helmholtz, in *On the Sensation of Tone* (Dover, New York, 1954), Chaps. I and XVI (originally published 1885).

[3]J. Douthett, R. Entringer, and A. Mullhaupt, "Musical Scale Construction: The Continued Fraction Comporomise," Utilitas Mathematica **42**, 97–113 (1992).

[4]M. Schechter, "Tempered Scales and Continued Fractions," Am. Math. Monthly **87**, 40–42 (1980).

[5]A. Ya Khinchin, *Continued Fractions* (The University of Chicago Press, Chicago, 1992).

[6]R. J. Krantz and J. Douthett, "A Measure of the Reasonableness of Equal-Tempered Musical Scales," J. Acoust. Soc. Am. **95** (6), 3642–3650 (1994).

[7]D. E. Hall, "Acoustical Numerology and Lucky Temperaments," Am. J. Phys. **56** (4), 329–333 (1988).

Richard J. Krantz
Metropolitan State College of Denver
Department of Physics
Denver, Colorado 80217

Answer to Question #34. What is the third law of thermodynamics trying to tell us?

The question posed by Blau and Halfpap[1] has previously been addressed by Landsberg[2] in an authoritative answer based on thermodynamic arguments. Rather than pursue the thermodynamic route again, we would like to elaborate more on some of the microscopic aspects raised in the question presentation[1] since the third law is better understood in those cases where molecular information on the system under consideration is available. In this way, we are addressing the general title of the original question rather than particular aspects of the two questions raised by the authors.[1] It is our belief that not only the *answers* but also the *questions* are relevant in the understanding of the third law.

The elementary molecular models invoked in our statistical physics courses show that $S/Nk \ll 1$ for $T/T_{ch} \ll 1$ provided that we use the quantum rather than the classical description, where S is the entropy of the N particles system, k is Boltzmann's constant, T is the temperature, and T_{ch} is a finite, characteristic temperature that can be obtained in terms of the particular system properties. For a system of non-interacting distinguishable spins in a magnetic field modeling a paramagnetic solid, T_{ch} could be 1 K. For an ideal Bose gas simulating some qualitative features of liquid helium, $T_{ch} \approx 1$ K. (For a photon gas, however, both S and N

go to zero as T^3, and then $S/Nk = 3.6$ regardless of the temperature of the system.) For an ideal Fermi gas representing the free electron gas in a metal, $T_{ch} \approx 10^4$ K. Finally, for an ideal gas of diatomic molecules, T_{ch} can take different values within a very large range of temperatures depending on the degree of freedom probed experimentally. Most of the above temperatures are significantly higher than zero, and reflect the properties of the spectrum of states available to the particles of the system and the average number of particles in each state.

Statements like "the third law tells us that the entropy of a pure substance is zero at 0 K" and "at 0 K the system should be in its lowest energy (ground) state and then $S = 0$ according to Boltzmann's law" are certainly very vivid and useful, but might as well be supplemented with more empirical views. What both experimental data and the molecular models used to explain these data tell us is that the entropy of a system becomes exceedingly small for temperatures lower than T_{ch}, where T_{ch} is a finite temperature that depends on the particular system considered.[3] Also, the number of states accessible for a macroscopic system is given by $\exp(S/k)$, and this number can still be relatively large compared to unity[4] even when the entropy is negligible and $S/k \ll N \approx 10^{24}$.

Historically, the third law was put forward at the beginning of this century and thus no reference to quantum aspects was made. We have emphasized above that some crucial characteristics of the third law arise naturally only when the molecular information available for the system under consideration is used consistently within the quantum statistical physics framework. However, since one of the salient features of thermodynamics is its independence from particular microscopic models, the molecular information is not invoked in this physical science. In this context, the third law must be postulated as a separate law in order to explain the experimental data gathered for all thermodynamic systems at low enough temperatures.

[1]S. Blau and B. Halfpap, "Question #34. What is the third law of thermodynamics trying to tell us?," Am. J. Phys. **64** (1), 13–14 (1996).

[2]P. T. Landsberg, "Answer to Question #34. What is the third law of thermodynamics trying to tell us?," Am. J. Phys. **65** (4), 269–270 (1997).

[3]F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill, New York, 1962), Chap. 3.

[4]See, e.g., G. V. Rosser, *An Introduction to Statistical Physics* (Ellis Horwood, Chichester, 1982), Chap. 3 and problem 3.1.

Salvador Mafé and Juan de la Rubia
Depto. de Termodinámica
Facultad de Física
Universidad de Valencia
E-46100 Burjassot, Spain

Answer to Question #51. Applications of third-order and fifth-order differential equations

Neuenschwander[1] asks about physics applications of third-order and fifth-order differential equations. It was, of course, Newton's great insight that the causes of changes in motion affect the second derivative of position. So it is rare that a time derivative higher than the second occurs in the differ-

ential equation of motion of a physical system. The well-known example of the use of a third time derivative is in the electromagnetic radiation reaction. See, for example, Erik J. Bochove, "Unified derivation of classical radiation forces," Am. J. Phys. **64** (11), 1419–1422 (1996) (the same issue of the *Journal* in which the Question appeared!).

[1]Dwight E. Neuenschwander, "Question 51. Applications of third-order and fifth-order differential equations," Am. J. Phys. **64** (11), 1353 (1996).

Kirk T. McDonald
Joseph Henry Laboratories
Department of Physics
Princeton University
P.O. Box 708
Princeton, New Jersey 08544

Answer to Question #51. Applications of third-order and fifth-order differential equations

In regard to physics applications of higher odd-order differential equations about which Neuenschwander inquires [Am. J. Phys. **64** (11), 1353 (1996)], they are not quite as rare as we think. In calculating the flow along a flat plate at zero incidence in a uniform stream, although the boundary layer equation is of the second order, the separated differential equation is the famous Blasius equation, $f''' + ff''$.[1] We get a system of three third-order equations in a similar problem in boundary layer flows and wake problems in magnetofluid dynamics, one of which is the Blasius equation.[2]

Of more interest is the fact that a third-order partial differential equation, the famous Korteweg-de Vries equation $u_t + uu_t + u_{xxx} = 0$, is derived in the physical context of water surface gravity waves and ion acoustic waves in plasmas.[3] A fifth-order KdV equation appears in evolutionary equations (using Schrödinger operators) in soliton theory.[4] A list of higher-order equations appears in Zwillinger's *Handbook of Differential Equations*.[5]

Sophus Lie's "Theorie der Transformationsgruppen" connects differential equations with group theory.[6] To every linear differential equation of order n corresponds a finite group of linear homogeneous transformations on n variables, which has properties similar to those of the group of permutations for an algebraic equation. There is a similarity here with Felix Klein's "Erlanger Programm" in geometry. A geometry is categorized by the properties that remain invariant under a given group of transformations. Similarly group theory determines all the possible forms which are permissible for the field equations due to the geometric relationship between the geometric properties of space with the sources.[7] Unfortunately, I can't find a current reference to show how a given group of the physical transformation forces mostly second-order partial differential equations.

[1]L. Rosenhead, *Laminar Boundary Layers* (Dover, New York, 1988), pp. 222–223.

[2]S. Pai, *Modern Fluid Mechanics* (Science Press, New York, 1981), pp. 302–304.

[3]E. Infeld and G. Rowlands, *Nonlinear Waves, Solitons and Chaos* (Cambridge U.P., Cambridge, 1990), p. 7.

[4]P. G. Drazin and R. S. Johnson, *Solitons, an Introduction* (Cambridge U.P., Cambridge, 1989), p. 102.

[5]D. Zwillinger, *Handbook of Differential Equations* (Academic, Boston, 1989), pp. 128–129, 132–136.

[6]I. N. Sneddon, *Encyclopaedic Dictionary of Mathematics for Engineers and Applied Scientists* (Pergamon, Oxford, 1976), pp. 428–431.

[7]E. T. Bell, *Development of Mathematics* (McGraw-Hill, New York, 1945), 2nd ed., pp. 436 and 443.

George DeRise
Department of Mathematics
Thomas Nelson Community College
Hampton, Virginia 23670
deriseg@tncc.cc.va.us

Answer to Question #62. When did the indeterminacy principle become the uncertainty principle?

The answer is: almost right from the beginning. It was called the "uncertainty principle" long before Peter Bergmann's 1942 text[1] cited by William D. Harris.[2] Use of the name "indeterminacy principle," as found in Bergmann's text, has been infrequent in the United States since as early as 1929.

The story has several interesting twists. In the March 1927 paper[3] in which he first derived what we usually call the uncertainty relations, Heisenberg never used the German word for "indeterminacy," *Unbestimmtheit*. He referred instead to the *Ungenauigkeit*, inexactness, in measurements of canonically conjugate variables in quantum mechanics. However, in an appendix added to the page proofs of the paper, in which he refers to further discussions with Niels Bohr, Heisenberg did use the word *Unsicherheit*, uncertainty, for the first time, but it was also the only time he used "uncertainty" in that period.

Although Bohr and Heisenberg spoke Danish together, since Heisenberg was fluent in Danish by 1927, Bohr was apparently the source of the word "uncertainty" as used then by Heisenberg (in equivalent German) and ever since then in English-language literature. This is suggested by a paper[4] Bohr presented in English six months later to an international meeting of physicists at Lake Como, Italy, commemorating the centenary of the death of Alessandro Volta. The paper reached English-language audiences in *Nature* soon afterward. Here Bohr used the word "uncertainty" throughout, and in a recently published English draft manuscript related to this lecture Bohr referred to Heisenberg's inequalities for the first time as the "uncertainty relations."[5] In remarks[6] in German following Bohr's Lake Como lecture Heisenberg used instead the word *Ungenauigkeitsrelation*, which in English is the rather awkward "inexactness relation."

Aside from not using the German word for "indeterminacy" in his March 1927 paper, Heisenberg did not refer to these relations as forming the basis for a new "principle." As far as I can establish, the words "indeterminacy" and "principle" did not appear, separately or together, in this connection until two years later, but ironically they fostered the more common English-language usage "uncertainty principle," rather than "indeterminacy principle." Heisenberg used the word *Unbestimmtheitsrelationen*, indeterminacy relations, for the first time in a German survey[7] of the history of quantum theory written just before he left Germany for a tour of America and the Far East from March to November 1929. This word appeared again in the German manuscript for a series of lectures that Heisenberg delivered