The relation between expressions for time-dependent electromagnetic fields given by Jefimenko and by Panofsky and Phillips

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The expressions of Jefimenko, which have received much recent attention in this Journal, are contained in Sec. 14.3 of the book *Classical Electricity and Magnetism* by Panofsky and Phillips. The latter develop these expressions further into a form that gives greater emphasis to the radiation fields. This article presents a derivation of the various expressions and discusses an apparent paradox in applying Panofsky and Phillips's result to static situations.

I. INTRODUCTION

A general method of calculation of time-dependent electromagnetic fields was given by L. Lorenz in 1867,[1] in which the retarded potentials were first introduced. These are

$$\phi(\mathbf{x},t)=\int \frac{[\rho(\mathbf{x}',t')]}{R}\,d\mathbf{x}',\qquad \mathbf{A}(\mathbf{x},t)=\frac{1}{c}\int \frac{[\mathbf{j}(\mathbf{x}',t')]}{R}\,d\mathbf{x}', \tag{1}$$

where ϕ and $\mathbf{A}$ are the scalar and vector potentials in Gaussian units, ρ and $\mathbf{j}$ are the charge and current densities, $R=|\mathbf{R}|$ with $\mathbf{R}=\mathbf{x}-\mathbf{x}'$, and a pair of brackets, [ ], implies the quantity within is to be evaluated at the retarded time $t'=t-R/c$ with c being the speed of light. Lorenz did not explicitly display the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$, although he noted they could be obtained via

$$\mathbf{E}=-\nabla\phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t},\quad \mathbf{B}=\nabla\times\mathbf{A}. \tag{2}$$

Had Lorenz's work been better received by Maxwell, the expressions discussed below probably would have been well known over a century ago. But Maxwell's struggles to provide a model for time-dependent electric and magnetic fields left him insufficient energy to grapple with the prospect that time-dependent potentials were useful tools (and hence deserved models of their own). The retarded potentials came into general use only after Hertz's experiments on electromagnetic waves (1888) and Thomson's discovery of the electron (1895). At this time basic interest switched from electromagnetic phenomena due to time-dependent charge and current distributions to that due to moving electrons, i.e., point charges. Hence the Liénard–Wiechert potentials and the corresponding expressions for the electromagnetic fields of a point charge in arbitrary motion form the basis for most subsequent discussions.

For historical perspectives see the books of Whittaker[2] and O'Rahilly.[3] The textbook by Becker[4] contains concise derivations very much in the spirit of the original literature (and is still in print).

Recent interest shown in this Journal in general expressions for time-dependent electromagnetic fields arose from an article by Griffiths and Heald[5] on the conundrum: while time-dependent potentials are ''simply'' the retarded forms of the static potentials, the time-dependent fields are more than the retarded forms of the Coulomb and the Biot–Savart laws. Of course, it was Maxwell who first expounded the resolution of the conundrum; the something extra is radiation! Hertz's great theoretical paper on electric-dipole radiation (especially the figures) remains the classic example of how time-dependent fields can be thought of as instantaneous static fields close to the source but as radiation fields far from the source.[6]

The discussion of Griffiths and Heald centered on the following expressions for the electromagnetic fields, which they attributed to Jefimenko:[7]

$$\mathbf{E}=\int \frac{[\rho]\hat{\mathbf{n}}}{R^2}\,d\mathbf{x}'+\frac{1}{c}\int \frac{[\dot{\rho}]\hat{\mathbf{n}}}{R}\,d\mathbf{x}'-\frac{1}{c^2}\int \frac{[\dot{\mathbf{j}}]}{R}\,d\mathbf{x}', \tag{3}$$

where $\dot{\mathbf{j}}=\partial\mathbf{j}/\partial t$, $\hat{\mathbf{n}}=\mathbf{R}/R$, and

$$\mathbf{B}=\frac{1}{c}\int \frac{[\mathbf{j}]\times\hat{\mathbf{n}}}{R^2}\,d\mathbf{x}'+\frac{1}{c^2}\int \frac{[\dot{\mathbf{j}}]\times\hat{\mathbf{n}}}{R}\,d\mathbf{x}'. \tag{4}$$

These expressions indeed contain retarded versions of the Coulomb and Biot–Savart laws as their leading terms, but their relation to radiation is not as manifest as it might be. In particular, Eq. (3) seems to suggest that there exist both longitudinal and transverse components of the electric field that fall off as $1/R$. It must be that the second term of Eq. (3) cancels the longitudinal component of the third term, although this is not self-evident. Thus, from a pedagogical point of view, Eq. (3) goes only part way toward resolving the conundrum.

Personally, I found some of the discussion by Griffiths and Heald (and their followers[8–13]) surprising in that I imagined it was common knowledge that Eq. (3) can be transformed to

$$\mathbf{E}=\int \frac{[\rho]\hat{\mathbf{n}}}{R^2}\,d\mathbf{x}'+\frac{1}{c}\int \frac{([\mathbf{j}]\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}+([\mathbf{j}]\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}}{R^2}\,d\mathbf{x}'+\frac{1}{c^2}\int \frac{([\dot{\mathbf{j}}]\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}}{R}\,d\mathbf{x}'. \tag{5}$$

The combination of Eqs. (4) and (5) manifestly displays the mutually transverse character of the radiation fields (those that vary as $1/R$), and to my taste better serves to illustrate the nature of the time-dependent fields. However, after extensive checking the only reference to Eq. (5) that I have located is in Sec. 14.3 of the 2nd edition of the textbook of Panofsky and Phillips.[14,15]

The alert reader may be troubled by the second term in Eq. (5), which seems to suggest that static currents give rise to an electric field. One can verify by explicit calculation that this is not so for current in a straight wire or (more tediously) in a circular loop. Indeed, the second term in Eq. (5) vanishes whenever both $\nabla\cdot\mathbf{j}=0$ and $\dot{\mathbf{j}}=0$ over the whole current distribution, i.e., in the static limit.

In the following section I give a direct derivation of Eq. (5) in possible contrast to that of Panofsky and Phillips who used Fourier transforms. Section III clarifies why the second term of Eq. (5) vanishes in the static limit.

II. DERIVATION OF THE ELECTRIC FIELD

The starting point is, of course, Eq. (2) applied to Eq. (1). The time derivative $\partial/\partial t$ acts only on $[\mathbf{j}]$ because of the relation $t'=t-R/c$. Thus

$$-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}=-\frac{1}{c^2}\int\frac{[\dot{\mathbf{j}}]}{R}\,d\mathbf{x}'. \tag{6}$$

Also,

$$-\nabla\phi=-\int\nabla\left(\frac{[\rho]}{R}\right)d\mathbf{x}'=-\int[\rho]\nabla\left(\frac{1}{R}\right)d\mathbf{x}'-\int\frac{\nabla[\rho]}{R}\,d\mathbf{x}'. \tag{7}$$

But

$$\nabla\left(\frac{1}{R}\right)=-\frac{\hat{\mathbf{n}}}{R^2},\qquad \nabla[\rho]=\nabla\rho(\mathbf{x}',t-R/c)=[\dot{\rho}]\left(-\frac{\nabla R}{c}\right)=-\frac{[\dot{\rho}]\hat{\mathbf{n}}}{c}. \tag{8}$$

Equations (6)–(8) combine to give Eq. (3).

It is now desired to transform the second term of Eq. (3), and the continuity equation,

$$\nabla\cdot\mathbf{j}=-\dot{\rho}, \tag{9}$$

suggests itself for this purpose. Some care is needed to apply this at the retarded coordinates $\mathbf{x}'$ and $t'=t-R/c$ because of the implicit dependence of the current density on $\mathbf{x}'$ through R. Introducing $\nabla'=\partial/\partial\mathbf{x}'$, then

$$\nabla'\cdot[\mathbf{j}]=[\nabla'\cdot\mathbf{j}]+[\dot{\mathbf{j}}]\cdot\left(-\frac{\nabla' R}{c}\right)=-[\dot{\rho}]+\frac{[\dot{\mathbf{j}}]\cdot\hat{\mathbf{n}}}{c}. \tag{10}$$

Thus

$$\frac{1}{c}\int\frac{[\dot{\rho}]\hat{\mathbf{n}}}{R}\,d\mathbf{x}'=-\frac{1}{c}\int\frac{(\nabla'\cdot[\mathbf{j}])\hat{\mathbf{n}}}{R}\,d\mathbf{x}'+\frac{1}{c^2}\int\frac{([\dot{\mathbf{j}}]\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}}{R}\,d\mathbf{x}'. \tag{11}$$

If the first term on the right-hand side of Eq. (11) actually varies as $1/R^2$, then the radiation field within Eq. (3) will have the form given in Eq. (5), since the last term in Eq. (11) is the negative of the longitudinal component of the last term in Eq. (3).

The integral involving $\nabla'\cdot[\mathbf{j}]$ can be transformed further by examining the components of the integrand:

$$\frac{(\nabla'\cdot[\mathbf{j}])\hat{\mathbf{n}}_i}{R}=\frac{\partial[\mathbf{j}]_j}{\partial x_j'}\frac{R_i}{R^2}=\frac{\partial}{\partial x_j'}\left(\frac{[\mathbf{j}]_jR_i}{R^2}\right)-[\mathbf{j}]_j\frac{\partial}{\partial x_j'}\left(\frac{R_i}{R^2}\right)=\frac{\partial}{\partial x_j'}\left(\frac{[\mathbf{j}]_jR_i}{R^2}\right)+\frac{[\mathbf{j}]_i-2([\mathbf{j}]\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}_i}{R^2}, \tag{12}$$

where summation is implied over index j. The volume integral of the first term becomes a surface integral with the aid of Gauss's theorem, and hence vanishes assuming the currents are contained within a bounded volume:

$$\int_V\frac{\partial}{\partial x_j'}\left(\frac{[\mathbf{j}]_jR_i}{R^2}\right)d\mathbf{x}'=\int_S d\mathbf{S}'\cdot\left(\frac{[\mathbf{j}]R_i}{R^2}\right)=0. \tag{13}$$

The remaining term can be summarized as

$$-\frac{1}{c}\int\frac{(\nabla'\cdot[\mathbf{j}])\hat{\mathbf{n}}}{R}\,d\mathbf{x}'=\frac{1}{c}\int\frac{2([\mathbf{j}]\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}-[\mathbf{j}]}{R^2}\,d\mathbf{x}'=\frac{1}{c}\int\frac{([\mathbf{j}]\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}+([\mathbf{j}]\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}}{R^2}\,d\mathbf{x}'. \tag{14}$$

Finally, Eqs. (3), (11), and (14) combine to yield Eq. (5).

III. THE STATIC LIMIT

To ascertain the behavior of the second term of Eq. (5) in the static limit, refer to Eq. (14), which indicates its relation to $\nabla'\cdot[\mathbf{j}]$. The latter is expanded in Eq. (10), and accordingly vanishes if both $\nabla\cdot\mathbf{j}=0$ and $\dot{\mathbf{j}}=0$. Since $\nabla\cdot\mathbf{j}=-\dot{\rho}$, the second term of Eq. (5) vanishes in the static limit (i.e., when both $\dot{\rho}$ and $\dot{\mathbf{j}}$ vanish), as claimed in Sec. I.

It remains that expression (5) may be more cumbersome than expression (3) for explicit calculation of the electric field in time-dependent situations where radiation is not the dominant concern. This point has been illustrated in the calculation of the fields of a moving charge[5,8] and related examples.[10,11,16–18] In another article[19] I discuss how Eqs. (3) and (4) can be used to clarify a subtle issue regarding the fields outside an infinite solenoid with a time-dependent current.

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