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Damping of Radial Pinch Effects

Abstract

Calculations indicate that the radial pinch effect as a mercury jet enters a 20-T solenoid magnet is strongly damped with time constant $4\rho c^2/\sigma B_0^2 \approx 10^{-4}$ s, and less than one percent of the initial kinetic energy of the jet is lost to Joule heating. Any disruption of the jet due to the radial pinch will likely be localized to the tail of the jet, which may fall slightly behind the bulk motion.

1 Estimate of Radial Motion of a Ring of Mercury

We consider a ring of mercury of radius r that is inside a jet which is moving with velocity v_z along the axis of a solenoid magnet, whose axial field is $B_z(z)$.

The motion of the jet results in an apparent time dependence to the magnetic field in the jet's frame, which leads to eddy currents $\mathbf{J}$, and hence to $\mathbf{J}\times\mathbf{B}$ forces that can distort the jet. Of course, the magnetic field cannot change the energy of the mercury, and the Ohmic losses of the eddy currents (and viscous damping of shear flow of the mercury) decrease the kinetic energy of the jet. So in general, the jet will slow down as it passes through the magnetic field, and the velocity reduction will be a function of position within the jet.

The magnetic flux $\Phi = \pi r^2 B_z$ through the ring varies with time because the ring is moving through the spatially varying field B_z, and because the radius of the ring is varying at rate $v_r = dr/dt$. So, an azimuthal electric field E_ϕ is induced around the ring:

$$2\pi r E_\phi = -\frac{1}{c}\frac{d\Phi}{dt} = -\frac{\pi r^2}{c}\frac{dB_z}{dt} - \frac{2\pi r v_r B_z}{c} = -\frac{\pi r^2 v_z}{c}\frac{\partial B_z}{\partial z} - \frac{2\pi r v_r B_z}{c}. \tag{1}$$

This electric field leads to an azimuthal current density

$$J_\phi = \sigma E_\phi = -\frac{\sigma r v_z}{2c}\frac{\partial B_z}{\partial z} - \frac{\sigma v_r B_z}{c}, \tag{2}$$

where σ ($\approx 10^{16}$/s in Gaussian units) is the conductivity of mercury ($\sigma_{\rm Hg} \approx \sigma_{\rm Cu}/60$). This azimuthal current interacts with the longitudinal magnetic field to produce a radial force density,

$$f_r = \frac{J_\phi B_z}{c} = -\frac{\sigma r v_z B_z}{2c^2}\frac{\partial B_z}{\partial z} - \frac{\sigma v_r B_z^2}{c^2}. \tag{3}$$

The first term is negative as the mercury enters the strong-field region, and we speak of a radial pinch. The second term arises only if the mercury takes on a radial velocity due to the radial pinch, and then it opposes the pinch effect.

What is the result of this force? Since mercury is essentially incompressible, it cannot simply implode radially. Any radial motion must be accompanied by longitudinal motion; the pinch will tend to extrude the jet into a longer, thinner form, possibly leading to breakup of the jet. The extrusion will not be out the front of the jet, as the $\mathbf{J} \times \mathbf{B}_r$ force opposes this. The extrusion will be largely to the rear of the jet.

Here, we don't offer a model of the extrusion process, but we do estimate the effect of the second term of eq. (3), which damps the extrusion process.

For a first estimate, we approximate the longitudinal velocity v_z as constant, which is reasonable so long as the radial velocity v_r remains small compared to this.

We completely ignore the inertial effect of the rest of the jet on the ring that will further oppose any radial shrinkage of the jet. We simply suppose that the ring deforms longitudinally without requiring any work, and write a radial equation of motion as

$$f_r = \rho \frac{dv_r}{dt}, \tag{4}$$

where $\rho = 13.6$ g/cm^3 is the mass density of mercury. Then,

$$\frac{dv_r}{dt} = -\frac{\sigma r v_z B_z}{2\rho c^2} \frac{\partial B_z}{\partial z} - \frac{\sigma v_r B_z^2}{\rho c^2}. \tag{5}$$

If the solenoid magnet has inner diameter d and central field strength B_0, the peak value of $\partial B_z/\partial z$ is about B_0/d, which occurs in the fringe field where $B_z \approx B_0/2$. So, the radial acceleration can be estimated as

$$\ddot{r} = \frac{dv_r}{dt} \approx -\frac{\sigma v_z B_0^2}{4\rho c^2 d} r - \frac{\sigma B_0^2}{4\rho c^2} \dot{r} = -\frac{\sigma B_0^2}{4\rho c^2} \left(\frac{v_z}{d} r + \dot{r} \right). \tag{6}$$

This equation holds for time interval d/v_d during which the ring passes through the fringe field of the magnet.

For $B_0 = 20$ T $= 2 \times 10^5$ G, the magnetic damping time in mercury,

$$\tau = \frac{4\rho c^2}{\sigma B_0^2}, \tag{7}$$

has the value $4 \cdot 13.6 \cdot (3 \times 10^{10})^2 / 10^{16} \cdot (2 \times 10^5)^2 \approx 1.2 \times 10^{-4}$ sec.

If $v_z \approx 20$ m/s $= 2000$ cm/s, and the magnetic aperture is $d \approx 20$ cm, then $d/v_z = 0.01$ s $= 100\,\tau$. That is, the radial pinch occurs over a time scale 100 times larger than the damping time. Hence, the radial acceleration is quickly damped to a very small value, and the radial velocity is given by eq. (6) as

$$\dot{r} = -\frac{v_z}{d} r. \tag{8}$$

Then, the radius of the ring obeys

$$r = r_0 e^{-v_z t/d}. \tag{9}$$

This equation applies for time $t \approx d/v_d$, so we predict that the radius of the jet shrinks by $1/e$ as it enters the magnet.

We also see that the radial acceleration is roughly $(v_z/d)^2 r$, which is about $(v_z/d)/(\sigma B_0^2/4\rho c^2) \approx 0.01$ times that which would occur if there were no damping term in eq. (6).

The reduction in radius of the ring is accompanied by a longitudinal stretch. Naively, if the ring were inside a jet that was 10 cm long initially, a pinch to 1/3 of its initial radius would require the jet to stretch to about 100 cm long. The longitudinal expansion of 90 cm would occur in time 0.01 sec, so the longitudinal velocity required for this, relative to the initial velocity v_z, is 9000 cm/s = 90 m/s. But this is larger than our assumed initial value of v_z, and the implication is that the tail of the jet would be repulsed by the magnetic and accelerated to greater than its initial energy.

2 Consistency with Conservation of Energy

To see whether this conclusion is reasonable, we recall that any reduction in the kinetic energy of the jet is due to Joule heating losses (when we ignore viscosity, as is the case here). If we simply insert the approximate results (8) and (9) in eq. (2), we would have $j \approx 0$, and there would be no energy loss to drive any changes in the velocity of the jet.

We must be more careful. As a guide, we note that the kinetic energy density in the jet is

$$\mathrm{ke} = \frac{\rho}{2}(v_r^2 + v_z^2), \tag{10}$$

whose time rate of change is

$$\frac{d\mathrm{ke}}{dt} = \rho \left(v_r \frac{dv_r}{dt} + v_z \frac{dv_z}{dt} \right). \tag{11}$$

This should be equal to minus the rate of Joule heating per unit volume,

$$\begin{aligned} -\frac{du_{\mathrm{Joule}}}{dt} &= -\frac{j^2}{\sigma} = -\sigma \left(\frac{r v_z}{2c} \frac{\partial B_z}{\partial z} + \frac{v_r B_z}{c} \right)^2 \\ &= -v_r \left(\frac{\sigma r v_z B_z}{2c^2} \frac{\partial B_z}{\partial z} + \frac{\sigma v_r B_z^2}{c^2} \right) - v_z \left[\frac{\sigma r v_r B_z}{2c^2} \frac{\partial B_z}{\partial z} + \frac{\sigma r^2 v_z}{4c^2} \left(\frac{\partial B_z}{\partial z} \right)^2 \right], \end{aligned} \tag{12}$$

using eq. (2). Comparing with eq. (5), we see that the first term is indeed $\rho v_r dv_r/dt$, so the second must be $\rho v_z dv_z/dt$. Indeed, the longitudinal $\mathbf{j} \times \mathbf{B}$ force is

$$\rho \frac{dv_z}{dt} = f_z = \frac{j_\phi B_r}{c} \approx -\frac{j_\phi r}{2c} \frac{\partial B_z}{\partial z} = -\frac{\sigma r v_r B_z}{2c^2} \frac{\partial B_z}{\partial z} - \frac{\sigma r^2 v_z}{4c^2} \left(\frac{\partial B_z}{\partial z} \right)^2, \tag{13}$$

using eq. (2).

Consistency with energy conservation requires that we deal with the radial and longitudinal equations of motion (5) and (13) at the same time. Further, these two equations apply only to an isolated ring of mercury. For an extended jet, these equations of motion must be embedded in the Naiver-Stokes equation that describes the fluid dynamics of the mercury.

3 Combined Analysis of Radial and Longitudinal Motion of a Ring

Here, we continue to ignore viscosity, but pursue the simultaneous solution of eqs. (5) and (13). For this, we present estimates based on the analytic form of the axial magnetic field for a semi-infinite solenoid, as discussed in sec. 2.5 of [1]. The solenoid coil is taken to be a thin cylindrical sheet of radius $d/2$, extending from $z = 0$ to $+\infty$. When this is used to approximate a physical solenoid of length $l = \alpha d$, the center of physical solenoid corresponds to $z = \alpha d/2$ in the semi-infinite model, whose results should only be used for z less than this value.

For a physical solenoid of central field B_0, the corresponding axial magnetic field of the semi-infinite solenoid is

$$B_z(0, z) = \frac{B_0}{2}\left(1 + \frac{z}{\sqrt{(d/2)^2 + z^2}}\right), \tag{14}$$

whose derivative is

$$B_z' = \frac{dB_z(0, z)}{dz} = \frac{B_0}{2}\frac{(d/2)^2}{[(d/2)^2 + z^2]^{3/2}}. \tag{15}$$

We also note from [1] that the longitudinal $\mathbf{j} \times \mathbf{B}$ force on the induced current in the ring leads to an equation of motion for the longitudinal velocity v_z of the jet,

$$v_z'(r) = -\frac{\sigma r^2 (B_z')^2}{4\rho c^2}. \tag{16}$$

Using eq. (14) in eq. (16) and integrating the equation of motion from $-\infty$ to z assuming that r is constant (*i.e.*, neglecting magnetic damping), we find that v_z is reduced by

$$\Delta v_z(r) = -\frac{3\pi\sigma r^2 B_0^2}{64\rho c^2 d}. \tag{17}$$

For the parameters used above, $\Delta v_z(1\ \mathrm{cm}) \approx -2.4$ m/s.

For the case when both v_r and v_z of the ring are time dependent, I integrated the equations of motion (2) and (13) using an EXCEL spreadsheet [2]. The results indicate that eqs. (8) and (9) are still fairly good, *i.e.*, the radius of the ring would like to shrink to about 1/3 its initial value as the ring enters the solenoid. But as the radius shrinks, the longitudinal force (13) is reduced, and the calculation indicates almost no change in the longitudinal velocity v_z. The Joule losses are calculated to be only about 0.002 of the initial kinetic energy.

These results could well apply to an isolated ring. As the radius of the ring shrinks, its longitudinal extent would grow to keep the volume constant, but the average longitudinal velocity could remain essentially constant.

This behavior is, however, not consistent with the ring being part of an extended jet (tube) of mercury. The ring cannot readily distort longitudinally because the surrounding material also is trying to expand longitudinally. This will result in a longitudinal pressure that can largely suppress any longitudinal motion of the mercury relative to its average velocity.

At the back of the jet, this internal pressure is unopposed by the air or vacuum, and the tail of the jet may tend to disperse somewhat. However, the calculation indicates that there is very little energy available to drive this dispersal, and should the mercury start to move longitudinally relative to the initial v_z, it will be subject to additional magnetic damping. At the front of the jet, the longitudinal magnetic forces will almost certainly keep the jet from dispersing. Hence, I predict that the distortion of the jet due to the radial pinch is modest.

Recall also that an ANSYS model [3] by C. Lu that included the inertial effects showed quite small effects due to the radial pinch (Fig. 1).

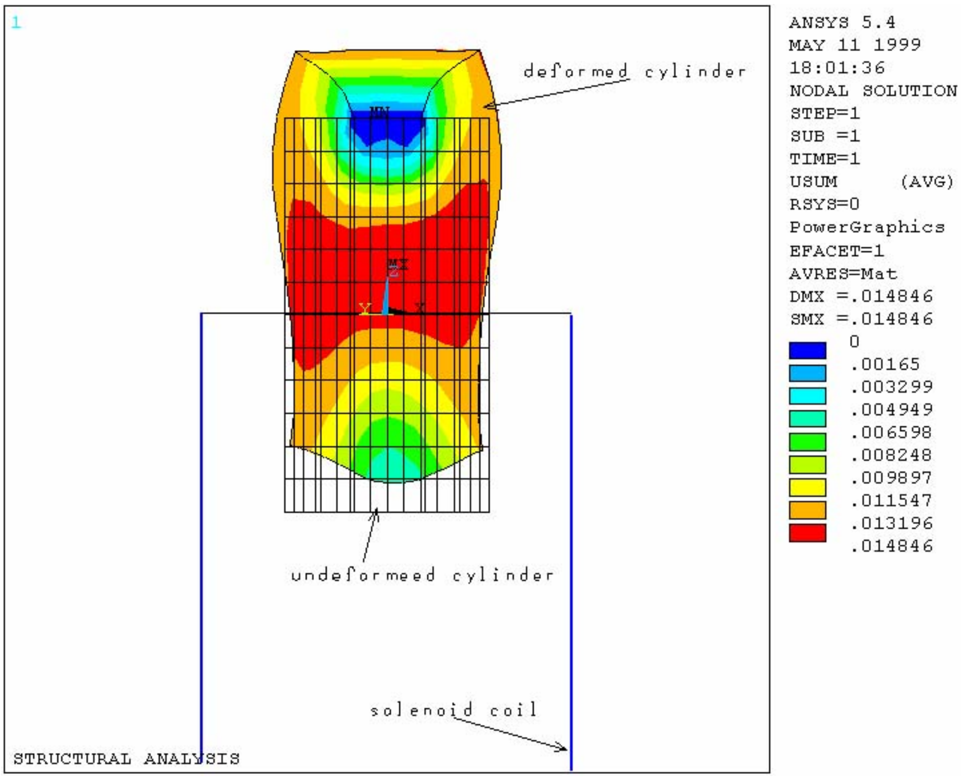

Figure 1: ANSYS model of a mercury jet entering a 20-T solenoid magnet [3].

A Appendix: The Dipole Energy $-\boldsymbol{\mu} \cdot \mathbf{B}$

Ralf Prigl has raised the interesting question as to the relation between the preceding energy analysis and the energy

$$U = -\boldsymbol{\mu} \cdot \mathbf{B} \qquad (18)$$

of a magnetic dipole $\boldsymbol{\mu}$ in an external magnetic field.

The expression (18) is deduced in considerations of a fixed magnetic moment that moves so slowly in a magnetic field that induction effects may be neglected. It is not clear that it has meaning in the present case.

The quantity U is a configuration energy, meaning that if the geometric configuration of the system changes work must be done corresponding to the change in U. In particular, the force acting on the dipole can be derived from U according to $\mathbf{F} = -\boldsymbol{\nabla} U$.

Recall that a magnetic field cannot do work on a charged particle, so conversely a charged particle cannot do work on a magnetic field. Hence, we should not think of U as being stored in the magnetic field.[1] Rather, energy U is stored in the external agent that holds the dipole in the magnetic field.[2]

The present problem is not one of magnetostatics, and there is no external agent interacting mechanically with the mercury. If eq. (18) is to have any meaning, it would have to refer to the kinetic energy and/or the thermal energy of the mercury.

To explore the possible meaning of eq. (18) we consider the case of a solid metal ring of fixed radius r. We restrict the discussion to rings coaxial with the magnet. Then, when the ring is carrying current density J_ϕ, the magnitude of its magnetic moment is

$$\mu = \frac{IA}{c} = \frac{\pi r^2 I}{c} = \frac{\pi r^2 a J_\phi}{c} = \frac{r J_\phi \mathrm{Vol}}{2c}, \tag{19}$$

where a is the cross sectional area of the ring, whose volume is Vol $= \ 2\pi r a$. If the current is induced by longitudinal motion of the ring with velocity v_z, then from eq. (2) the magnitude of the current density is

$$J_\phi = \frac{\sigma r v_z}{2c} \frac{\partial B_z}{\partial z}, \tag{20}$$

and the magnetic moment is

$$\mu = \frac{\sigma r^2 v_z \mathrm{Vol}}{4c^2} \frac{\partial B_z}{\partial z}. \tag{21}$$

The magnetic moment vector is opposite to the magnetic field $\mathbf{B}$ that induces it, according to Lenz' law. Hence, the energy (18) is positive, with value

$$U = \frac{\sigma r^2 v_z B_z \mathrm{Vol}}{4c^2} \frac{\partial B_z}{\partial z}. \tag{22}$$

To compare with previous results, I cast this energy into the form of an energy density per unit volume of the ring:

$$u = \frac{U}{\mathrm{Vol}} = \frac{\sigma r^2 v_z B_z}{4c^2} \frac{\partial B_z}{\partial z}. \tag{23}$$

Let us now see whether the longitudinal force [eq. (13) with v_r set to zero] on the moving ring can be deduced from eq. (23):

$$f_z \stackrel{?}{=} -\frac{du}{dz} = -\frac{\sigma r^2}{4c^2} \left[\frac{dv_z}{dz} B_z \frac{\partial B_z}{\partial z} + v_z \left(\frac{\partial B_z}{\partial z} \right)^2 + v_z B_z \frac{\partial^2 B_z}{\partial z^2} \right]. \tag{24}$$

The middle term of eq. (24) is indeed the desired result for f_z, but the other two terms do not sum to zero, since $dv_z/dz = f_z/\rho v_z$.

I conclude that eq. (18) does not represent a meaningful energy for the present problem.

[1] Indeed, a simple calculation shows that the interaction energy $\int \mathbf{B}_{\mathrm{dipole}} \cdot \mathbf{B}_{\mathrm{external}} d\mathrm{Vol}/4\pi$ is zero for a dipole in a uniform external field, independent of the orientation of the dipole relative to the field.

[2] Recall Earnshaw's theorem that there is no equilibrium possible for a system of charges in a static electric or magnetic field – and that a dipole can be thought of as a pair of magnetic charges.

References

[1] J. Alessi *et al.*, *An R&D Program for Targetry and Capture at a Muon Collider Source* (Sept. 30, 1998, approved as BNL E951),
http://www.hep.princeton.edu/~mcdonald/mumu/target/targetprop.pdf

[2] The 1MB EXCEL spreadsheet can be accessed at
http://www.hep.princeton.edu/~mcdonald/mumu/target/radialpinch.xls
The main table has been truncated to 1000 lines to make the file more compact. It should be extended to about 7000 lines to calculate past the center of the solenoid.

[3] C. Lu, *ANSYS Coupled-Field Analysis in the Simulation of Liquid Metal Moving in the Magnetic Field*, Princeton/MuMu/99-24 (May 20, 1999),
http://puhep1.princeton.edu.mumu/target/liquidtarget.pdf