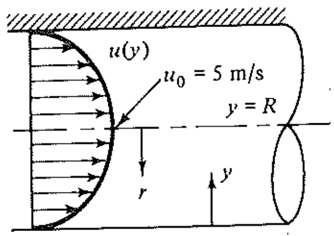


Fig. E6.3

This is a very small shear stress, but it will cause a large pressure drop in a long pipe (170 Pa for every 100 m of pipe).

(*c*) The average velocity V is found by integrating the logarithmic-law velocity distribution

$$V = \frac{Q}{A} = \frac{1}{\pi R^2} \int_0^R u 2\pi r \, dr \tag{2}$$

Introducing $u = u^*[1/\kappa \ln (yu^*/\nu) + B]$ from Eq. (6.21) and noting that $y = R - r$, we can carry out the integration of Eq. (2), which is rather laborious. The final result is

$$V = 0.835 u_0 = 4.17 \text{ m/s} \qquad \textit{Ans. (c)}$$

We shall not bother showing the integration here because it is all worked out and a very neat formula is given in Eqs. (6.49) and (6.59).

Notice that we started from almost nothing (the pipe diameter and the centerline velocity) and found the answers without solving the differential equations of continuity and momentum. We just used the logarithmic-law, Eq. (6.21), which makes the differential equations unnecessary for pipe flow. This is a powerful technique, but you should remember that all we are doing is using an experimental velocity correlation to approximate the actual solution to the problem.

We should check the Reynolds number to ensure turbulent flow

$$\text{Re}_d = \frac{Vd}{\nu} = \frac{(4.17 \text{ m/s})(0.14 \text{ m})}{1.51 \times 10^{-5} \text{ m}^2/\text{s}} = 38{,}700$$

Since this is greater than 4000, the flow is definitely turbulent.

6.4 FLOW IN A CIRCULAR PIPE

As our first example of a specific viscous-flow analysis, we take the classic problem of flow in a full pipe, driven either by pressure or by gravity or both. Figure 6.10 shows the geometry of the pipe of radius R. The x axis is taken in the flow direction and is inclined to the horizontal at an angle ϕ.

Before proceeding with a solution to the equations of motion, we can learn a lot by making a control-volume analysis of the flow between sections 1 and 2 in Fig.

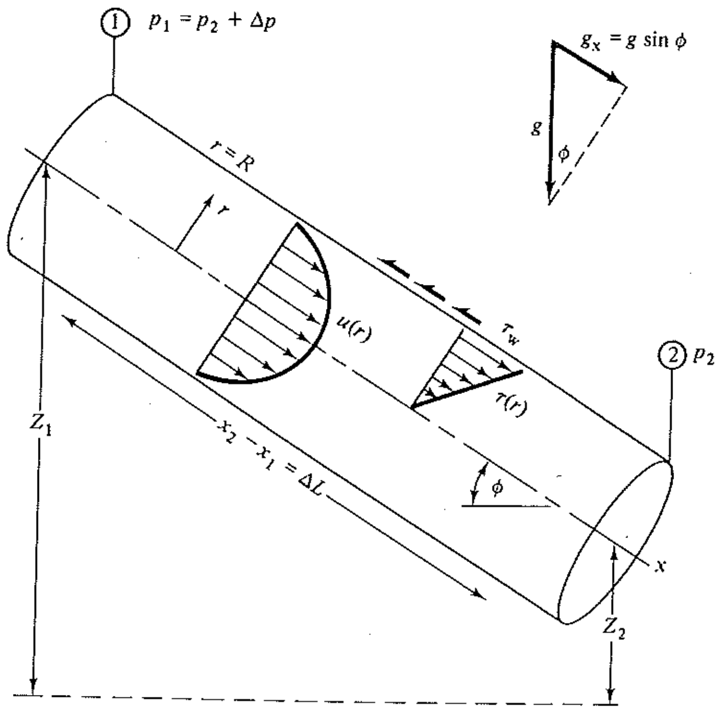


Fig. 6.10 Control volume of steady fully developed flow between two sections in an inclined pipe.

6.10. The continuity relation, Eq. (3.23), reduces to

$$Q_1 = Q_2 = \text{const}$$

or

$$V_1 = \frac{Q_1}{A_1} = V_2 = \frac{Q_2}{A_2} \tag{6.23}$$

since the pipe is of constant area. The steady-flow energy equation (3.85) reduces to

$$\frac{p_1}{\rho} + \tfrac{1}{2}\alpha_1 V_1^2 + gz_1 = \frac{p_2}{\rho} + \tfrac{1}{2}\alpha_2 V_2^2 + gz_2 + gh_f \tag{6.24}$$

since there are no shaft-work or heat-transfer effects. Now assume that the flow is fully developed (Fig. 6.6) and correct later for entrance effects. Then the kinetic-energy correction factor $\alpha_1 = \alpha_2$ and, since $V_1 = V_2$ from (6.23), Eq. (6.24) now reduces to a simple expression for the friction-head loss h_f

$$h_f = \left(z_1 + \frac{p_1}{\rho g}\right) - \left(z_2 + \frac{p_2}{\rho g}\right) = \Delta\left(z + \frac{p}{\rho g}\right) = \Delta z + \frac{\Delta p}{\rho g} \tag{6.25}$$

The pipe-head loss equals the change in the sum of pressure and gravity head, i.e., the change in height of the HGL. Since the velocity head is constant through the pipe, h_f also equals the height change of the EGL. Recall from Fig. 3.17 that the EGL decreases downstream in a flow with losses unless it passes through an energy source, e.g., as a pump or heat exchanger.

Finally apply the momentum relation (3.40) to the control volume in Fig. 6.10, accounting for applied forces due to pressure, gravity, and shear

$$\Delta p \, \pi R^2 + \rho g(\pi R^2) \, \Delta L \sin \phi - \tau_w(2\pi R) \, \Delta L = \dot{m}(V_1 - V_2) = 0 \tag{6.26}$$

This equation relates h_f to the wall shear stress

$$\Delta z + \frac{\Delta p}{\rho g} = h_f = \frac{2\tau_w}{\rho g} \frac{\Delta L}{R} \tag{6.27}$$

where we have substituted $\Delta z = \Delta L \sin \phi$ from Fig. 6.10.

So far we have not assumed either laminar or turbulent flow. If we can correlate τ_w with flow conditions, we have resolved the problem of head loss in pipe flow. Functionally, we can assume that

$$\tau_w = F(\rho, V, \mu, d, \epsilon) \tag{6.28}$$

where ϵ is the wall-roughness height. Then dimensional analysis tells us that

$$\frac{8\tau_w}{\rho V^2} = f = F\left(\mathrm{Re}_d, \frac{\epsilon}{d}\right) \tag{6.29}$$

The dimensionless parameter f is called the *Darcy friction factor*, after Henry Darcy (1803–1858), a French engineer whose pipe-flow experiments in 1857 first established the effect of roughness on pipe resistance.

Combining Eqs. (6.27) and (6.29), we obtain the desired expression for finding pipe-head loss

$$h_f = f \, \frac{L}{d} \frac{V^2}{2g} \tag{6.30}$$

This is the Darcy-Weisbach equation, valid for duct flows of any cross section and for laminar and turbulent flow. It was proposed by Julius Weisbach, a German professor who in 1850 published the first modern textbook on hydrodynamics.

Our only remaining problem is to find the form of the function F in Eq. (6.29) and plot it in the Moody chart of Fig. 6.13.

Equations of Motion

For either laminar or turbulent flow the continuity equation in cylindrical coordinates is given by (Appendix E)

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial u}{\partial z} = 0 \tag{6.31}$$

We assume that there is no swirl or circumferential variation, $v_\theta = \partial/\partial\theta = 0$, and fully developed flow: $u = u(r)$ only. Then Eq. (6.31) reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) = 0$$

or

$$rv_r = \text{const} \tag{6.32}$$

But at the wall, $r = R$, $v_r = 0$ (no slip); therefore (6.32) implies that $v_r = 0$ everywhere. Thus in fully developed flow there is only one velocity component, $u = u(r)$.

The momentum differential equation in cylindrical coordinates now reduces to

$$\rho u \frac{\partial u}{\partial z} = -\frac{dp}{dx} + \rho g_x + \frac{1}{r}\frac{\partial}{\partial r}(r\tau) \tag{6.33}$$

where τ can represent either laminar or turbulent shear. But the left-hand side vanishes because $u = u(r)$ only. Rearrange, noting from Fig. 6.10 that $g_x = g \sin\phi$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\tau) = \frac{d}{dx}(p - \rho g x \sin\phi) = \frac{d}{dx}(p + \rho g z) \tag{6.34}$$

Since the left-hand side varies only with r and the right-hand side varies only with x, it follows that both sides must be equal to the same constant.[1] Therefore we can integrate Eq. (6.34) to find the shear distribution across the pipe, utilizing the fact that $\tau = 0$ at $r = 0$

$$\tau = \tfrac{1}{2} r \frac{d}{dx}(p + \rho g z) = (\text{const})(r) \tag{6.35}$$

Thus the shear varies linearly from the centerline to the wall, for either laminar or turbulent flow. This is also shown in Fig. 6.10. At $r = R$, we have the wall shear

$$\tau_w = \tfrac{1}{2} R \frac{\Delta p + \rho g \,\Delta z}{\Delta L} \tag{6.36}$$

which is identical with our momentum relation (6.27). We can now complete our study of pipe flow by applying either laminar or turbulent assumptions to fill out Eq. (6.35).

Laminar-Flow Solution

Note in Eq. (6.35) that the HGL slope $d(p + \rho g z)/dx$ is *negative* because both pressure and height drop with x. For laminar flow, $\tau = \mu\, du/dr$, which we substitute in Eq. (6.35)

$$\mu \frac{du}{dr} = \tfrac{1}{2} r K \qquad K = \frac{d}{dx}(p + \rho g z) \tag{6.37}$$

[1] Ask your instructor to explain this to you if necessary.

Integrate once

$$u = \tfrac{1}{4}r^2 \frac{K}{\mu} + C_1 \tag{6.38}$$

The constant C_1 is evaluated from the no-slip condition at the wall: $u = 0$ at $r = R$

$$0 = \tfrac{1}{4}R^2 \frac{K}{\mu} + C_1 \tag{6.39}$$

or $C_1 = -\tfrac{1}{4}R^2 K/\mu$. Introduce into Eq. (6.38) to obtain the exact solution for laminar fully developed pipe flow

$$u = \frac{1}{4\mu}\left[-\frac{d}{dx}(p + \rho g z)\right](R^2 - r^2) \tag{6.40}$$

The laminar-flow profile is thus a paraboloid falling to zero at the wall and reaching a maximum at the axis

$$u_{\max} = \frac{R^2}{4\mu}\left[-\frac{d}{dx}(p + \rho g z)\right] \tag{6.41}$$

It resembles the sketch of $u(r)$ given in Fig. 6.10.

The laminar distribution (6.40) is called *Hagen-Poiseuille flow* to commemorate the experimental work of G. Hagen in 1839 and J. L. Poiseuille in 1840, both of whom established the pressure-drop law, Eq. (6.1). The first theoretical derivation of Eq. (6.40) was given independently by E. Hagenbach and by F. Neumann around 1859.

Other pipe-flow results follow immediately from Eq. (6.40). The volume flux is

$$Q = \int u\, dA = \int_0^R u_{\max}\left(1 - \frac{r^2}{R^2}\right) 2\pi r\, dr$$

$$= \tfrac{1}{2}u_{\max}\pi R^2 = \frac{\pi R^4}{8\mu}\left[-\frac{d}{dx}(p + \rho g z)\right] \tag{6.42}$$

Thus the average velocity in laminar flow is one-half the maximum velocity

$$V = \frac{Q}{A} = \frac{Q}{\pi R^2} = \tfrac{1}{2}u_{\max} \tag{6.43}$$

For a horizontal tube ($\Delta z = 0$), Eq. (6.42) is of the form predicted by Hagen's experiment, Eq. (6.1)

$$\Delta p = \frac{8\mu L Q}{\pi R^4} \tag{6.44}$$

The wall shear is computed from the wall velocity gradient

$$\tau_w = \left| \mu \frac{du}{dr} \right|_{r=R} = \frac{2\mu u_{max}}{R} = \tfrac{1}{2}R \left| \frac{d}{dx}(p + \rho g z) \right| \tag{6.45}$$

This gives an exact theory for laminar Darcy friction factor

$$f = \frac{8\tau_w}{\rho V^2} = \frac{8(8\mu V/d)}{\rho V^2} = \frac{64\mu}{\rho V d}$$

or

$$f_{lam} = \frac{64}{\mathrm{Re}_d} \tag{6.46}$$

This is plotted in the Moody chart, Fig. 6.13. The fact that f drops off with increasing Re_d should not mislead us into thinking that shear decreases with velocity: Eq. (6.45) clearly shows that τ_w is proportional to u_{max} and, interestingly, independent of density because the fluid acceleration is zero.

The laminar head loss follows from Eq. (6.30)

$$h_{f,lam} = \frac{64\mu}{\rho V d} \frac{L}{d} \frac{V^2}{2g} = \frac{32\mu L V}{\rho g d^2} = \frac{128\mu L Q}{\pi \rho g d^4} \tag{6.47}$$

We see that laminar head loss is proportional to V.

EXAMPLE 6.4 An oil with $\rho = 900$ kg/m³ and $\nu = 0.0002$ m²/s flows upward through an inclined pipe as shown. The pressure and elevation are known at sections 1 and 2, 10 m apart. Assuming steady laminar flow, (*a*) verify that the flow is up, (*b*) compute h_f between 1 and 2, and compute (*c*) Q, (*d*) V, and (*e*) Re_d. Is the flow really laminar?

solution (*a*) For later use, calculate

$$\mu = \rho\nu = (900 \text{ kg/m}^3)(0.0002 \text{ m}^2/\text{s}) = 0.18 \text{ kg/(m} \cdot \text{s)}$$

$$z_2 = \Delta L \sin 40° = (10 \text{ m})(0.643) = 6.43 \text{ m}$$

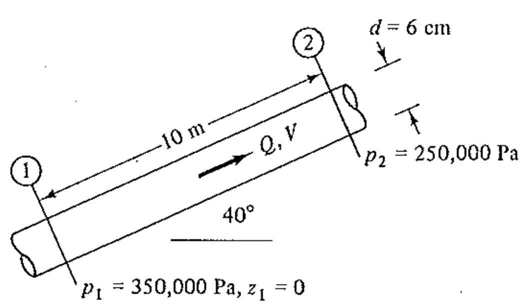


Fig. E6.4

The flow goes in the direction of falling HGL; therefore compute the grade-line height at each section

$$\text{HGL}_1 = z_1 + \frac{p_1}{\rho g} = 0 + \frac{350{,}000}{900(9.807)} = 39.65 \text{ m}$$

$$\text{HGL}_2 = z_2 + \frac{p_2}{\rho g} = 6.43 + \frac{250{,}000}{900(9.807)} = 34.75 \text{ m}$$

The HGL is lower at section 2; hence the flow is from 1 to 2 as assumed. *Ans.* (*a*)

(*b*) The head loss is the change in HGL height

$$h_f = \text{HGL}_1 - \text{HGL}_2 = 39.65 \text{ m} - 34.75 \text{ m} = 4.9 \text{ m} \qquad \textit{Ans. (b)}$$

Half the length of the pipe is quite a large head loss.

(*c*) We can compute Q from various laminar-flow formulas, notably Eq. (6.47)

$$Q = \frac{\pi \rho g d^4 h_f}{128 \mu L} = \frac{\pi(900)(9.807)(0.06)^4(4.9)}{128(0.18)(10)} = 0.0076 \text{ m}^3/\text{s} \qquad \textit{Ans. (c)}$$

(*d*) Divide Q by the pipe area to get the average velocity

$$V = \frac{Q}{\pi R^2} = \frac{0.0076}{\pi(0.03)^2} = 2.7 \text{ m/s} \qquad \textit{Ans. (d)}$$

(*e*) With V known, the Reynolds number is

$$\text{Re}_d = \frac{Vd}{\nu} = \frac{2.7(0.06)}{0.0002} = 810 \qquad \textit{Ans. (e)}$$

This is well below the transition value $\text{Re}_d = 2300$, and so we are fairly certain the flow is laminar.

Notice that by sticking entirely to consistent SI units (meters, seconds, kilograms, newtons) for all variables, no conversion factors whatever are needed in the calculations.

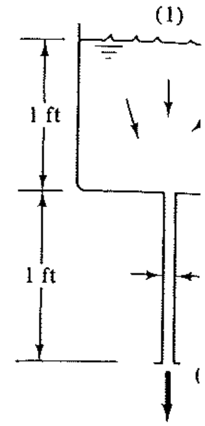
EXAMPLE 6.5 A liquid of specific weight $\rho g = 58 \text{ lb/ft}^3$ flows by gravity through a 1-ft tank and a 1-ft capillary tube at a rate of $0.15 \text{ ft}^3/\text{h}$, as shown. Sections 1 and 2 are at atmospheric pressure. Neglecting entrance effects, compute the viscosity of the liquid in slugs per foot-second.

solution Apply the steady-flow energy equation (3.86) with no heat transfer or shaft work

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \left(\frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2\right) + h_f$$

But $p_1 = p_2 = p_a$, and V_1 is negligible. Therefore, approximately,

$$h_f = z_1 - z_2 - \frac{V_2^2}{2g} = 2 \text{ ft} - \frac{V_2^2}{2g} \qquad (1)$$



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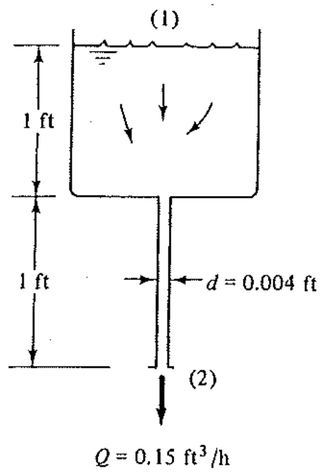

Fig. E6.5

But V_2 can be computed from the known volume flux and pipe diameter

$$V_2 = \frac{Q}{\pi R^2} = \frac{0.15/3600 \text{ ft}^3/\text{s}}{\pi(0.002 \text{ ft})^2} = 3.32 \text{ ft/s}$$

Substitution into Eq. (1) gives the net head loss

$$h_f = 2.0 - \frac{(3.32)^2}{2(32.2)} = 1.83 \text{ ft} \qquad (2)$$

Note that h_f includes the entire 2-ft drop through the system and not just the 1 ft of capillary pipe length.

Up to this point we have not specified laminar or turbulent flow. For laminar flow with negligible entrance loss, the head loss is given by Eq. (6.47)

$$h_f = 1.83 \text{ ft} = \frac{32\mu LV}{\rho g d^2} = \frac{32\mu(1.0 \text{ ft})(3.32 \text{ ft/s})}{(58 \text{ lb/ft}^3)(0.004 \text{ ft})^2} = 114{,}500\mu$$

or

$$\mu = \frac{1.83}{114{,}500} = 1.60 \times 10^{-5} \text{ slug/(ft} \cdot \text{s)} \qquad \textit{Ans.}$$

Note that L in this formula is the *pipe* length of 1 ft. Check the Reynolds number to see whether it is really laminar flow

$$\rho = \frac{\rho g}{g} = \frac{58.0}{32.2} = 1.80 \text{ slugs/ft}^3$$

$$\text{Re}_d = \frac{\rho V d}{\mu} = \frac{(1.80)(3.32)(0.004)}{1.60 \times 10^{-5}} = 1500 \qquad \text{laminar}$$

Since this is less than 2300, we seem to verify that the flow is laminar. Actually, we may be quite wrong, as Example 6.8 will show.