Chicane Orbit/Transport Equations

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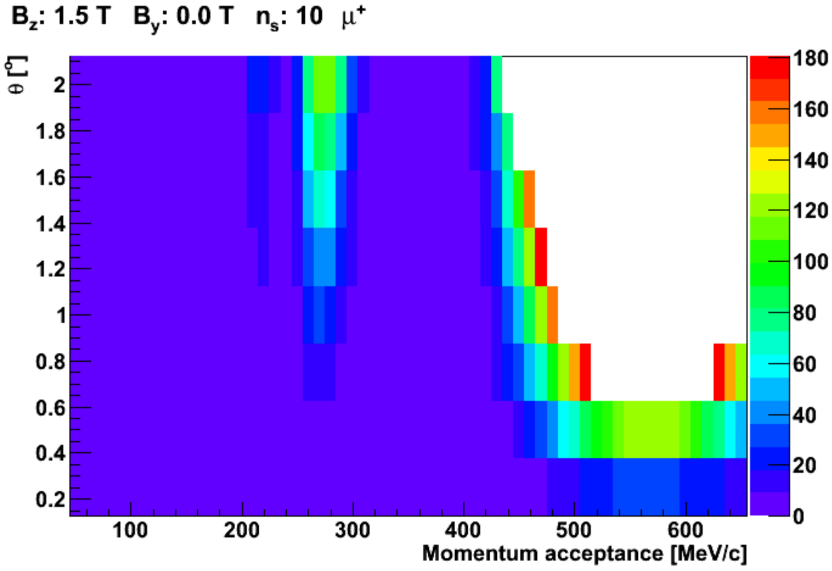
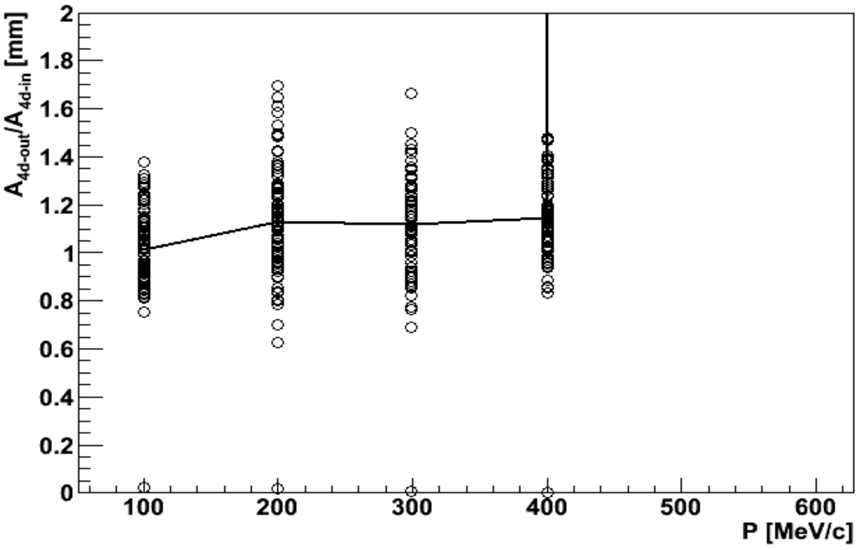
Bent solenoid chicane

- Try to understand
  - Why does the bent solenoid chicane work?
  - Why does it produce such a good match?
  - Why does it sometimes not produce a good match?
- So far, only look at constant radius, constant field chicane
  - i.e. not end sections or joins

Bent solenoid chicane





- Why, in general, do particles come back onto the axis after traversing the chicane?
- Why do some particles not come back on axis after traversing the chicane?
- Why does the beam overall seem well matched when it does come back to the axis?

3.1 Magnetic Field in a Constant Radius Constant Field Bent Solenoid

The magnetic field in a bent solenoid is assumed to have only a radial dependence, such that it can be written as

$$\vec{B}_{bs} = f(\rho)\vec{s} \tag{1}$$

From Maxwell's equations, in the absence of current sources

$$\begin{aligned}\vec{\nabla} \times \vec{B} \quad = \quad & (\frac{1}{\rho}\partial_s B_y - \partial_y B_s)\vec{\rho} + \\ & (\partial_y B_\rho - \partial_\rho B_y)\vec{s} + \\ & \frac{1}{\rho}(\partial_\rho \rho B_s - \partial_s B_\rho)\vec{y} = 0.\end{aligned}$$

Substituting for $\vec{B}_{bs}$ gives

$$\partial_\rho \rho f(\rho) = 0 \tag{2}$$

which has the solution

$$f(\rho) = \frac{b_0}{\rho} = \frac{b_s \rho_0}{\rho}. \tag{3}$$

Here b_s is the magnetic field strength on the reference orbit at radius ρ_0.

3.2 Helical Motion

In the presence of a field of this nature, some particles can be shown to travel in a helix. Starting from the Lorentz equations,

$$\vec{F} = \frac{d\vec{p}}{dt} = q\vec{v} \times \vec{B} \tag{4}$$

it is possible to derive the criterion for helical motion. Assume no radial velocity, so that

$$\vec{v} = c\frac{p_y\vec{y} + p_s\vec{s}}{E} \tag{5}$$

with speed of light c. Then if the particle is travelling at radius ρ

$$\vec{F} = qc\frac{p_y b_0}{E\rho} \tag{6}$$

For circular or helical motion, with constant energy,

$$\vec{F} = m\gamma\rho\omega^2 = \frac{m\gamma\beta_s^2 c^2}{\rho} = \frac{c^2 p_s^2}{E r_0}. \tag{7}$$

By equating the two expressions for $\vec{F}$

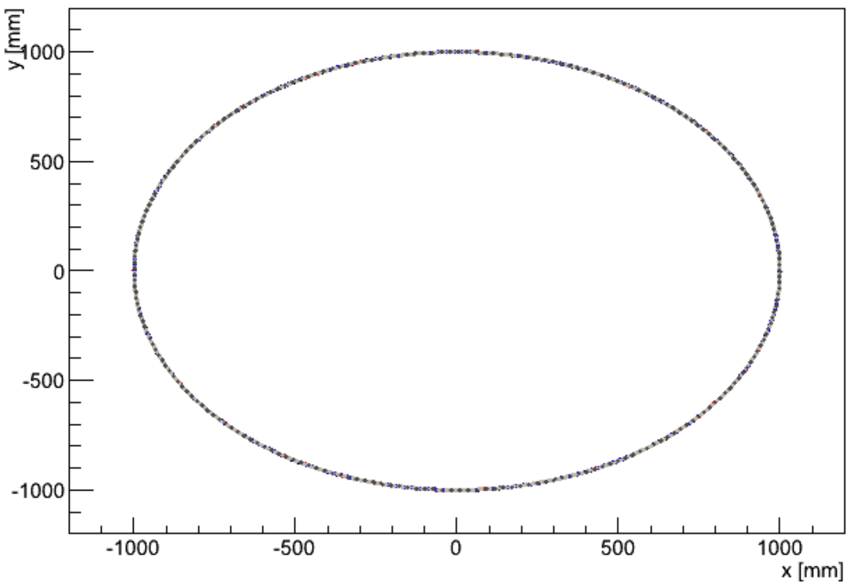
$$\frac{c^2 p_s^2}{E\rho} = qc\frac{p_y b_0}{E\rho}. \tag{8}$$

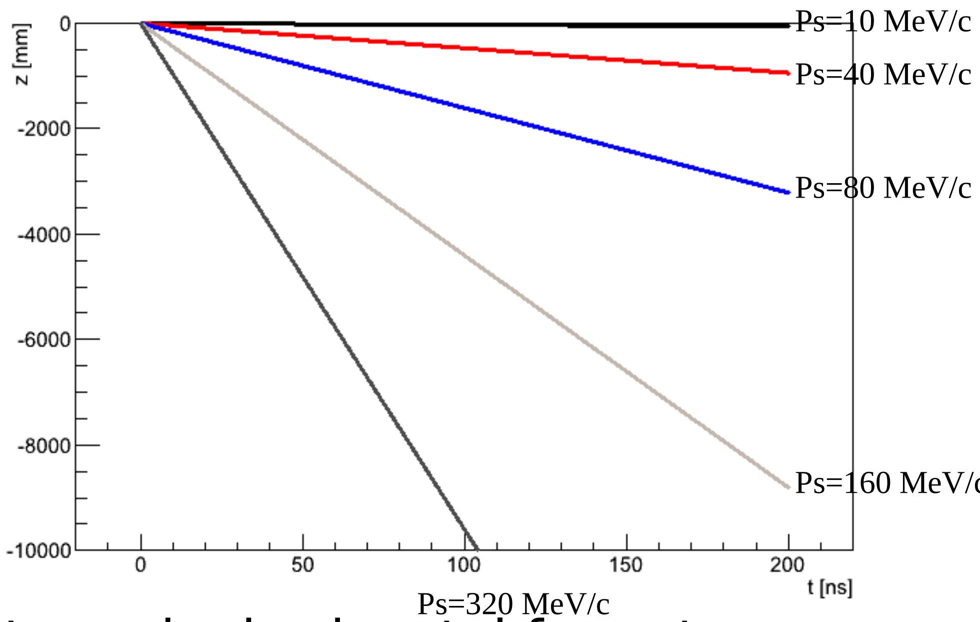
Then if

$$p_y = \frac{q p_s^2}{b_0} \tag{9}$$

particle motion will be on a helix. It should be noted that the slope of the helix is independent of the radius.

Helical motion





- Particles following the orbit need a horizontal force to push them back in the circle
- Get this by giving a vertical velocity –
  - cross product gives a horizontal force
- Bigger momentum means restoring force needs to be bigger
  - So bigger vertical velocity

Recipe for Transfer Matrix

- Extract vector potential
- Build Hamiltonian
- Expand Hamiltonian as a Taylor series
- Take Poisson bracket
- Gives transfer matrix

Vector Potential

The vector potential for the field can be found using the defining formula

$$\vec{B} = \nabla \times \vec{A} = \frac{b_0}{\rho}\vec{s} \tag{10}$$

Expanding the curl,

$$\begin{aligned} \partial_y A_\rho - \partial_\rho A_y &= \frac{b_0}{\rho} & (11) \\ \partial_s A_y - \rho \partial_\rho A_y &= 0 & (12) \\ \partial_\rho \rho A_s - \partial_s A_\rho &= 0 & (13) \end{aligned}$$

The gauge is chosen by comparison with the straight solenoid case,

$$\begin{aligned} A_\rho &= b_s y \frac{\rho_0}{\rho} - \frac{b_s y}{2} & (14) \\ A_s &= 0 & (15) \\ A_y &= -b_s \frac{\rho - \rho_0}{2} & (16) \end{aligned}$$

Vector Potential (cont.)

Near to the reference radius, with $\rho = \rho_0 + x$,

$$
\begin{aligned}
A_x &= b_s y \frac{\rho_0}{\rho_0 + x} - \frac{b_s y}{2} && (17) \\
&= b_s y \sum_{i=0}^{i=\infty} \left(-\frac{x}{\rho_0}\right)^i - \frac{b_s y}{2} && (18) \\
&= b_s y \sum_{i=1}^{i=\infty} \left(-\frac{x}{\rho_0}\right)^i + \frac{b_s y}{2} && (19) \\
A_s &= 0 && (20) \\
A_y &= -\frac{b_s x}{2}. && (21)
\end{aligned}
$$

It can be seen that as the radius of curvature tends to ∞, the vector potential tends to a straight solenoid potential, i.e.

$$
\lim_{\rho_0 \to \infty} A_x = \frac{b_s y}{2} \qquad (22)
$$

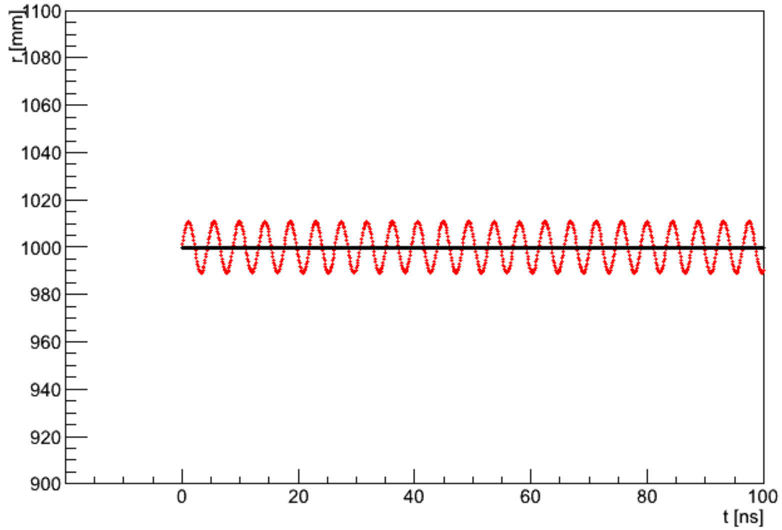
Cut a long story short...

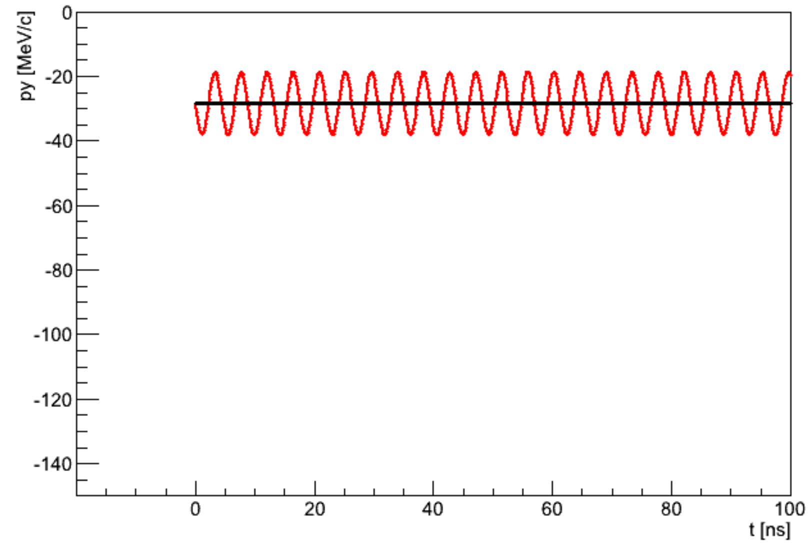
- In the small angle approximation terms ~ xy, x^2y, etc get lost
- End up with a “normal” solenoid vector potential
- Recover the normal (canonical) solenoid transfer matrix

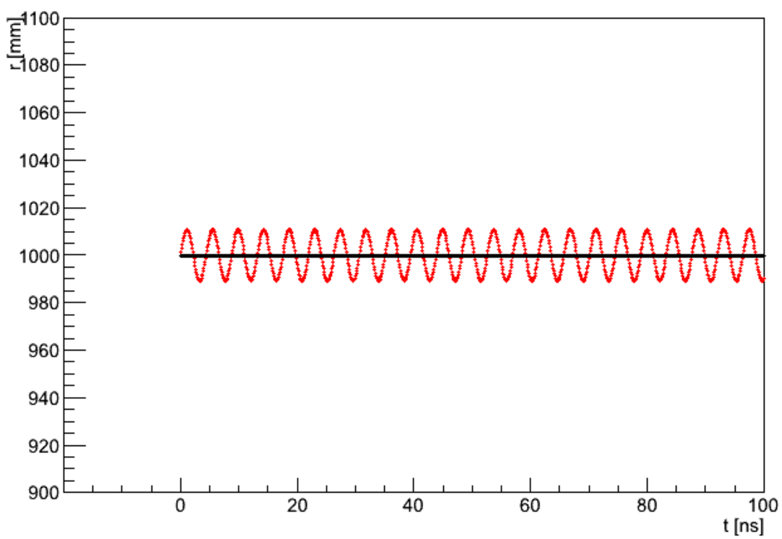
$$\begin{pmatrix} x \\ P_x \\ t \\ P_y \end{pmatrix}_{(s+ds)} = \begin{pmatrix} \frac{p}{\rho_0}dz \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & \frac{1}{p}dz & \frac{b_s}{2p}dz & 0 \\ \frac{-b_s^2}{4p}dz & 1 & 0 & \frac{b_s}{2p}dz \\ \frac{-b_s}{2p}dz & 0 & 1 & \frac{1}{p}dz \\ 0 & \frac{-b_s}{2p}dz & \frac{-b_s^2}{4p}dz & 1 \end{pmatrix} \begin{pmatrix} x \\ P_x \\ t \\ P_y \end{pmatrix}_{(s)}$$

- But note extra 0^{th} order term in x
  - Arises from curvature of coordinate system
- Can offset it by changing to helical reference as above
  - Borrow some -x from the transfer matrix
  - I have q = 1 here, and somehow a stray factor 2 … …
- Not quite there yet

Tracking



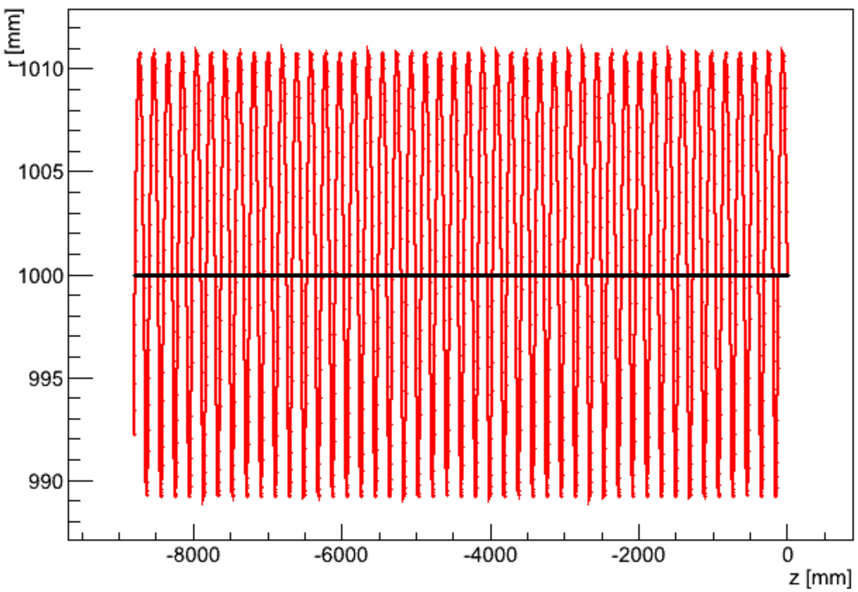




- Small kick in px
- Get very well behaved rotation about reference

Tracking


px [MeV/c]
10
5
0
-5
-10
-38
-36
-34
-32
-30
-28
-26
-24
-22
-20
-18
py [MeV/c]


r [mm]
1010
1005
1000
995
990
-8000
-6000
-4000
-2000
0
z [mm]

Conclusions

- It all looks quite nice so far
  - Still don't understand why chicane doesn't work sometimes
  - Time to look at the matching properties
  - i.e. Non-constant radius of curvature