

Yo-Yo Variants

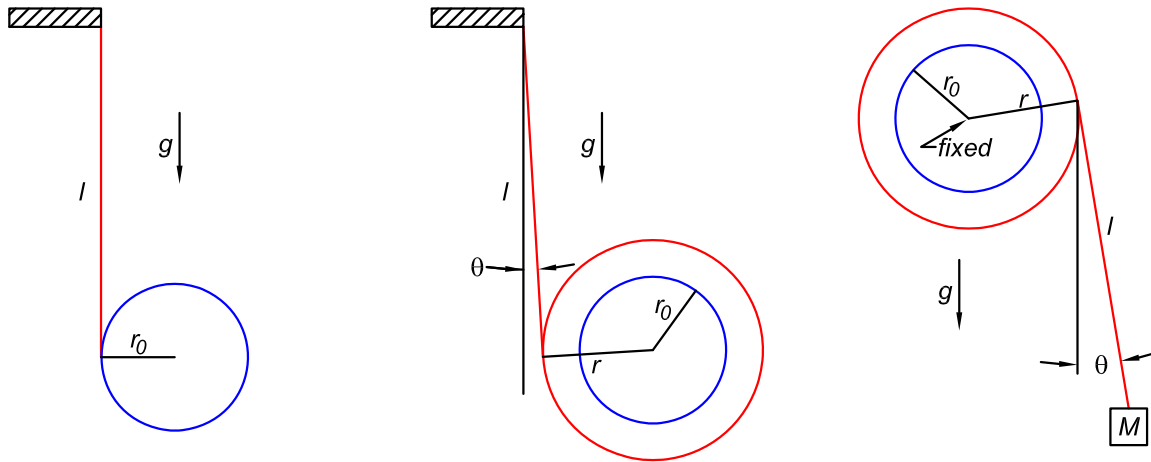
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1 Problem

In the classic yo-yo problem a spool of mass m , radius r_0 , and moment of inertia $I = kmr_0^2$ about its axis has a massless, infinitely thin string wrapped around the radius r_0 , with one end of the string fixed to a support above the spool, as shown in the left figure below.



Consider also the case that the string is a tape of length L , thickness t and mass per unit length ρ , with either the end of the tape fixed as in the middle figure above, or with the axis of the spool fixed and mass M attached to the end of the tape.

Discuss the motion in all three cases, assuming that the length l of the unwound portion of the string/tape is initially zero.

2 Solution

2.1 Massless String

The vertical speed v of the center of mass of the spool is related to its angular velocity ω by $v = \dot{l} = \omega r_0$. Hence, the (constant) energy of the yo-yo, which starts from rest with $l = 0$ at time $t = 0$, is,

$$E = 0 = \frac{mv^2}{2} + \frac{I\omega^2}{2} - mgl = \frac{m(1+k)\dot{l}^2}{2} - mgl. \quad (1)$$

Thus,

$$\dot{l}^2 = \frac{2gl}{1+k}, \quad \ddot{l} = \frac{g}{1+k}, \quad l = \frac{gt^2}{2(1+k)}. \quad (2)$$

For the case of a solid cylinder of radius r_0 , $k = 1/2$, $\ddot{l} = 2g/3$ and $l = gt^2/3$.

2.2 Yo-Yo with Massive Tape

We next consider a yo-yo suspended by a tape, rather than a string, of length L , thickness t , linear mass density ρ and mass $m_t = \rho L$. When length l of the tape has been unwound, the remaining tape on the yo-yo has area $(L - l)t = \pi(r^2 - r_0^2)$, where,

$$r = \sqrt{\frac{\pi r_0^2 + (L - l)t}{\pi}} \quad (3)$$

is the outer radius of the tape on the yo-yo. Then, the mass and moment of inertia of the yo-yo are,

$$m_y = m + \rho(L - l), \quad I_y = kmr_0^2 + \frac{\pi\rho(r^4 - r_0^4)}{2t}. \quad (4)$$

In the approximation that the length l of the unwound tape is straight, the system has only two degrees of freedom, which we take to be the length l and the angle θ of that length to the vertical. Then, taking the origin to be at the support point of the tape, the center of the yo-yo is at position,

$$x_y = l \sin \theta + r \cos \theta, \quad y_y = -l \cos \theta + r \sin \theta, \quad (5)$$

which point has velocity,

$$\dot{x}_y = \dot{l} \sin \theta + l \cos \theta \dot{\theta} + \dot{r} \cos \theta - r \sin \theta \dot{\theta}, \quad \dot{y}_y = -\dot{l} \cos \theta + l \sin \theta \dot{\theta} + \dot{r} \sin \theta + r \cos \theta \dot{\theta}. \quad (6)$$

The velocity of the point on the tape that is just about to lose contact with the winding, relative to the center of the yo-yo is $\dot{l} - r\dot{\theta}$, so the angular velocity of the yo-yo is,

$$\omega = \frac{\dot{l} - r\dot{\theta}}{r} = \frac{\dot{l}}{r} - \dot{\theta}. \quad (7)$$

The kinetic energy of the system is,

$$T = \frac{\rho l^3 \dot{\theta}^2}{6} + \frac{m_y}{2} \left(\dot{l}^2 + \dot{r}^2 + (l^2 + r^2) \dot{\theta}^2 + 2l\dot{r}\dot{\theta} - 2r\dot{l}\dot{\theta} \right) + \frac{I_y \omega^2}{2}, \quad (8)$$

and the potential energy is,

$$V = -\frac{\rho g l^2 \cos \theta}{2} + m_y g y_y. \quad (9)$$

Lagrange's method can now be used to deduce the equations of motion for coordinates l and θ , but these are somewhat complicated. Here, we content ourselves with yet another approximation, that angle θ is negligibly small. In this case, the kinetic energy of the system is,

$$T \approx \frac{m_y}{2} \left(\dot{l}^2 + \dot{r}^2 \right) + \frac{I_y \dot{l}^2}{2r^2} \quad (\theta = 0), \quad (10)$$

and the potential energy is,

$$V = -\frac{\rho g l^2}{2} - m_y g l \quad (\theta = 0). \quad (11)$$

This is still somewhat complicated, so we restrict our attention to the case of a roll of tape wound on itself, *i.e.*, $m = 0 = r_0$, for which,

$$r = \sqrt{\frac{(L-l)t}{\pi}}, \quad \dot{r} = -\frac{t\dot{l}}{2\sqrt{\pi(L-l)t}}, \quad m_y = \rho(L-l), \quad I_y = \frac{\pi\rho r^4}{2t}, \quad (12)$$

$$T \approx \frac{3\rho\dot{l}^2(L-l)}{4} + \frac{\rho\dot{l}^2 t}{8\pi} \quad (\theta = m = r_0 = 0), \quad (13)$$

and the potential energy is,

$$V = -\rho g l \left(L - \frac{l}{2} \right) \quad (\theta = m = r_0 = 0). \quad (14)$$

The total energy is zero, so $T = -V$, and,

$$\dot{l}^2 \left(1 + \frac{t}{6\pi(L-l)} \right) = \frac{2gl}{3} \frac{2L-l}{L-l} \quad (\theta = m = r_0 = 0). \quad (15)$$

Until the tape is almost completely unwound, $6\pi(L-l) \gg t$, during which time,

$$\dot{l}^2 \approx \frac{2gl}{3} \frac{2L-l}{L-l}, \quad \ddot{l} \approx g \frac{2L(L-l) + l^2}{3(L-l)^2} \quad (\theta = m = r_0 = 0). \quad (16)$$

The initial acceleration (when $l \approx 0$) is again $2g/3$, as found in sec. 2.1. The acceleration of the center of the roll of tape grows as it unwinds, reaching a maximum when $L-l \approx t$ with $\ddot{l}_{\max} \approx gL^2/3t^2 \gg g$.

Another approximation is that the tape is wound on a massless spool of radius r_0 and that the thickness t of the tape is negligible compared to r_0 . In this case the kinetic energy is $T = \rho(L-l)\dot{l}^2$ and the potential energy is $V = -\rho g l(L-l/2)$, such that,

$$\dot{l}^2 = g \frac{l(L-l/2)}{L-l}, \quad \ddot{l} = g \frac{2L(L-l) + l^2}{4(L-l)^2}. \quad (17)$$

Both \dot{l} and \ddot{l} diverge as l approaches L .

As discussed in [1, 2, 3, 4], if a roll of tape unwinds down an inclined plane, the end of the unrolling tape strikes the plane with very high speed, making a loud sound.

2.3 Unwinding Spool with Fixed Axle

For the third example sketched on p. 1, where the spool has a fixed axle and mass M hangs from the end of the tape, eqs. (3)-(7) hold again, but with the interpretation that (x_y, y_y) is the position of mass M . Then, (\dot{x}_y, \dot{y}_y) is the velocity of mass M , and of the lower end of the unwound tape. The velocity of the upper end of the unwound tape is the sum of the velocity of the point $\mathbf{p} = (r \cos \theta, r \sin \theta)$ where the tape leaves the spool and the velocity $\boldsymbol{\omega} \times \mathbf{p} = \omega(-p_y, p_x)$ of the tape relative to that point. That is,

$$\dot{x}_{\text{upper}} = \dot{p}_x - \omega p_y, \quad \dot{y}_{\text{upper}} = \dot{p}_y + \omega p_x, \quad (18)$$

such that the velocity of the center of mass of the unwound tape is,

$$\dot{x}_u = \frac{\dot{x}_y + \dot{x}_{\text{upper}}}{2}, \quad \dot{y}_u = \frac{\dot{y}_y + \dot{y}_{\text{upper}}}{2}. \quad (19)$$

The kinetic energy of this system is,

$$T = \frac{M}{2}(\dot{x}_y^2 + \dot{y}_y^2) + \frac{\rho l}{2}(\dot{x}_u^2 + \dot{y}_u^2) + \frac{\rho l^3}{24} \dot{\theta}^2 + \frac{I_y \omega^2}{2}, \quad (20)$$

Recalling that ρ is the linear mass density of the tape, the potential energy is,

$$V = Mgy_y + \rho lgy_u. \quad (21)$$

In the approximations that the angle θ of the tape to the vertical is negligible, and that terms in \dot{r} are negligible, then $\dot{\mathbf{p}} = 0$, $\omega = \dot{l}/r$, $\dot{x}_u = \dot{x}_y = 0$, $\dot{y}_u = \dot{y}_y = -\dot{l}$, and the kinetic energy becomes,

$$\begin{aligned} T &\approx \frac{\dot{l}^2}{2} \left(M + \rho l + \frac{I_y}{r^2} \right) \\ &= \frac{\dot{l}^2}{2} \left(M + \rho l + \frac{\pi k m r_0^2}{\pi r_0^2 + (L-l)t} + \frac{\rho[\pi r_0^2 + (L-l)t]}{2t} - \frac{\pi^2 \rho r_0^4}{2t[\pi r_0^2 + (L-l)t]} \right), \end{aligned} \quad (22)$$

while the potential energy is now,

$$V \approx -Mgl - \frac{\rho l^2 g}{2}. \quad (23)$$

The total energy is zero, so again $T = -V$, and,

$$\dot{l}^2 \left(M + \rho l + \frac{\pi k m r_0^2}{\pi r_0^2 + (L-l)t} + \frac{\rho[\pi r_0^2 + (L-l)t]}{2t} - \frac{\pi^2 \rho r_0^4}{2t[\pi r_0^2 + (L-l)t]} \right) = 2Mgl + \rho l^2 g. \quad (24)$$

The resulting expression for \dot{l} could be integrated numerically to find $l(t)$.

A somewhat trivial case is that the hanging mass M is large compared to the mass of the spool and tape, for which $\ddot{l} = g$. Another special case is that the spool has zero radius, $r_0 = 0$, such that,

$$\dot{l}^2 [2M + \rho(L+l)] = 2gl(2M + \rho l), \quad \ddot{l} = g \frac{2M[2M + \rho(L+2l)] + \rho^2 l(2L+l)}{[2M + \rho(L+l)]^2}. \quad (25)$$

If there is no hanging mass M , the acceleration of the unwound tape becomes,

$$j^2 = g \frac{2l^2}{L+l}, \quad \ddot{l} = g \frac{l(2L+l)}{(L+l)^2}, \quad (26)$$

which starts out at zero and rises to $3g/4$ when the tape is fully unwound.¹ The center of mass of the unwound tape is at,

$$y_{\text{cm}} = -\frac{l^2}{2L}, \quad \ddot{y}_{\text{cm}} = -\frac{l\ddot{l} + \dot{l}^2}{L} = -g \frac{l^2(4L+3l)}{L(L+l)^2}. \quad (27)$$

As the tape becomes fully unwound, the acceleration of the center of mass of the tape is $\ddot{y}_{\text{cm}} \rightarrow -7g/4$, but once the tape is fully unwound \ddot{y}_{cm} must be $-g$.²

Acknowledgment

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References

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¹The angular velocity of the still-wound tape diverges as the unwound length l approaches L , and the approximation that the unwound tape is entirely vertical breaks down.

²This discrepancy is further evidence that the approximations used here do not hold well as the length l of the unwound tape approaches L , for the idealized case that $M = 0$ and $r_0 = 0$.