Jefimenko's Wedge Circuit

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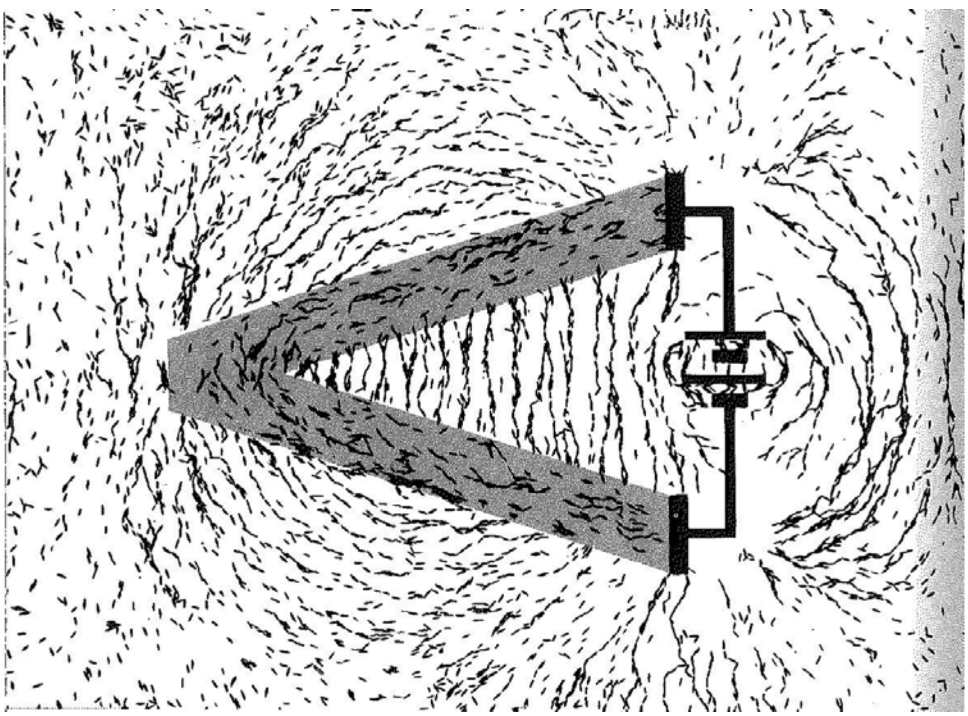
1 Problem

Electrical currents in resistive materials must be driven by an internal electric field $\mathbf{E} = \mathbf{J}/\varsigma$, where $\mathbf{J}$ is the current density and ς is the electrical conductivity. The internal electric field must be shaped by an appropriate density σ of surface electrical charge. In practice, the surface charge density is quite small, and is often neglected in discussions of electrical currents.[1] Some exceptions are [2]-[27].[2]

Jefimenko [7] considered the circuit illustrated below, in which two resistive plates of width r_0 are joined along one edge to form a wedge of angle 2α, with a battery of potential difference V_0 connected between the outer edges of the plates.



Deduce the potential, the electric field and the surface charge densities. Assuming the electrical current is I per unit length along the system (perpendicular to the plane of the

[1]This issue is distinct from the fact that a steady current of electrons in a conductor must have a tiny bulk charge density, of order v^2/c^2, where v is the velocity of the moving charges and c is the speed of light in vacuum. See, for example, [1].

[2]In [2] Sommerfeld deduced the electric field inside and outside of a wire that carries electromagnetic waves, but did not explicitly relate these fields to the surface charge density. Figure 5 of this paper is the source of Fig. 104 of [33].

figure), deduce the magnetic field and the Poynting vector [28].[3,4]

2 Solution

This two-dimensional problem is peculiar in that the potential is specified only on the plates, which by themselves do not form a complete boundary, yet standard procedures seem to "force" a solution that is appealing for the interior region of the circuit but nonsensical in the exterior region.

Away from the battery and conductors, the electric scalar potential V obeys Laplace's equation, $\nabla^2 V = 0$, for which a general solution to a two-dimensional potential in cylindrical coordinates (r, θ, z) is,[5]

$$V(r, \theta) = \sum (A_n \cos k_n\theta + B_n \sin k_n\theta)(C_n r^{k_n} + D_n r^{-k_n}) + (E + F\theta)(G + H \ln r). \tag{1}$$

Although the details of the battery and the wires that connect it to the outer edges of the plates have not yet been specified, these details partition the (r, θ) plane into two regions that we denote as the "exterior" and the "interior" of the circuit.

While the plates are symmetric about the x-axis, the battery and connecting leads need not be so. It seems only a minor restriction to assume that the potential V is antisymmetric in x and in θ. In this case, the potential is zero along the x-axis, *i.e.*, $V(r, \theta = 0, \pi) = 0$.

These "constraints" limit the form of the potential to,

$$V(r, \theta) = \begin{cases} \sum_n \sin k_n\theta (C_n r^{k_n} + D_n r^{-k_n}) + \theta(G + H \ln r) & \text{(interior)}, \\ \sum_n \sin n\theta (C'_n r^n + D'_n r^{-n}) & \text{(exterior)}, \end{cases} \tag{2}$$

noting that in the exterior solution, the parameters k_n must be integers n such that $\sin k_n\pi = 0$.

The boundary between the interior and exterior regions includes the plates, which lie on the planes $(r < r_0, \pm\alpha, z)$, where r_0 is the width of the plates and α is the half angle of the wedge. The potential should be continuous across these line segments, which appears to imply that the potential must have the form,

$$V(r, \theta) = \sum_n \sin n\theta (C_n r^n + D_n r^{-n}), \tag{3}$$

in both the exterior and interior regions.

[3]Poynting gave examples of his vector field for dc circuits, arguing that energy flows from the source (battery) to the load resistor across "empty" space (*i.e.*, through the electromagnetic field there) rather than inside the conductors that connect the two. This counterintuitive result often leads to skepticism, such as that in [29] (1897). Early comments such as [30, 31] did not perhaps settle the issue, while [2, 3, 4] represent the emergence of a consensus in support of Poynting's view.

[4]This problem inspired the possibly more straightforward variant of [14] (which also appeared in Fig. 2 of [4]). For discussion of this case by the author, see [32].

[5]See, for example, eq. (4-38) of [34], or eq. (7.100) of [35], which considers the present example in sec. 9.7.4.

Furthermore, the assumption of constant current in the resistive plates, whose outer edges are at potential $\pm V_0/2$, implies that the potential along the plates has the form,

$$V(r < r_0, \pm\alpha) = \pm \frac{V_0 r}{2r_0} . \tag{4}$$

This appears to force us to consider the two regions $r < r_0$ and $r > r_0$, such that index n can only be 1, and the potential is,

$$V(r,\theta) = \begin{cases} V_0 r \sin\theta / 2r_0 \sin\alpha = V_0 y / 2r_0 \sin\alpha & (r < r_0), \\ V_0 r_0 \sin\theta / 2r \sin\alpha & (r > r_0), \end{cases} \tag{5}$$

which is continuous at $r = r_0$ and satisfies the "boundary condition" (4). This form has the appealing implication that the electric field in the interior of the circuit is constant and in the y-direction,

$$\mathbf{E}_{\text{interior}} = -\frac{V_0}{2r_0 \sin\alpha} \hat{\mathbf{y}}, \tag{6}$$

in qualitative agreement with the figure on p. 1.[6]

However, the potential (5) also implies that eq. (6) applies in the exterior region of the circuit for $r < r_0$, and that the radial electric field is not continuous across the arc $(r_0, \alpha < |\theta| < \pi)$ where there is no material.[7]

That is, the arguments which led to the potential (5) cannot all be valid.

2.1 The Battery as a Boundary Condition

The previous discussion has been vague as to details about the battery, other than that its potential difference is V_0 and that the potential is assumed to be (anti)symmetric about the x-axis.

We could stipulate that the battery is consistent with the electric field (6) in various ways. A specification which emphasizes cylindrical coordinates is that the battery lies on the surface $(r_0, |\theta| < \alpha, z)$ with potential,

$$V_{\text{battery}}(r_0, |\theta| < \alpha, z) = \frac{V_0}{2} \frac{\sin\theta}{\sin\alpha} . \tag{7}$$

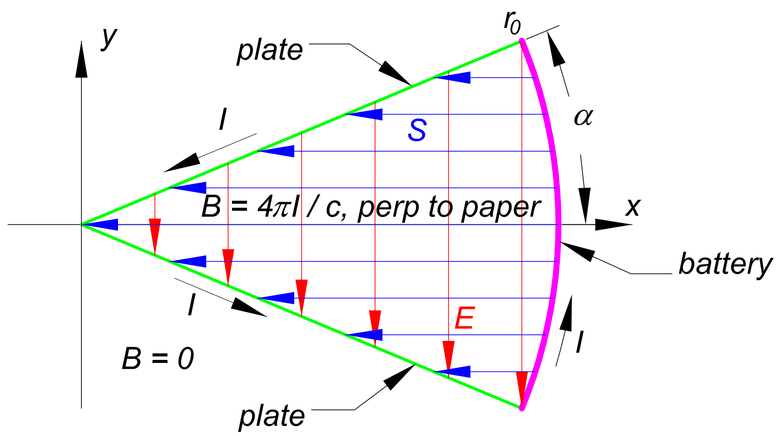
This is also consistent with the form (5) for the potential in both the exterior and interior $(r < r_0, |\theta| < \alpha)$ regions.

The potential (5) satisfies Laplace's equation in the interior region, but not in the exterior region since $\nabla^2 V \neq 0$ on the arc $(r_0, |\theta| > \alpha)$. Hence, we can conclude that the potential (5) is the unique solution in the interior region (since for this region it satisfies both Laplace's

[6]This form was also deduced by Jefimenko [7].

[7]The potential (5) is proportional to $\sin\theta/r$ at large r, as is reasonable for the present problem where the electric charges along the circuit (not yet deduced) have an electric dipole moment in the y-direction and hence an asymptotic potential of this form. That is, the potential (5) must be essentially correct at large r for all θ. *Thanks to Mark Heald for pointing this out.*

equation and the boundary conditions on the potential). However, the form (5) is not a valid solution in the exterior region.



It seems that a solution for the potential V via separation of variables in cylindrical coordinates, *i.e.*, eq. (1), does not exist in the exterior region. Similarly, if we were to consider use of rectangular coordinates, and a battery along the line ($x = r_0 \cos\alpha$, $|y| < r_0 \sin\alpha$) with potential $V_{\rm battery} = V_0 y/2$, we would find the same interior solution, but no solution via separation of variables in the exterior region.

Can a solution in the exterior region be found by some other means?

Since the problem is two-dimensional, there is an analytic function $f(t)$, where $t = x + iy$, whose real (or imaginary) part is the desired potential $V(x, y)$. But, can we find/guess the function f?

2.2 Surface Charges

Lacking an exterior solution we can only compute the surface charge density on the interior surface of the circuit.

Inside the conductors (and battery) the current and electric field are parallel to the surface, such that the normal component of the electric field inside the conductors is zero. Then, the surface charge density (per unit length, in Gaussian units) is given by,

$$\sigma_{\rm interior} = \frac{E_{\perp,{\rm interior}}}{4\pi} = \begin{cases} -\frac{E_{y,{\rm interior}}\cos\alpha}{4\pi} = \pm\frac{V_0}{8\pi\tan\alpha} & \text{(plates)}, \\ -\frac{E_{y,{\rm interior}}\sin\theta}{4\pi} = \frac{V_0\sin\theta}{8\pi\sin\alpha} & \text{(battery on } r = r_0\text{)}, \\ 0 & \text{(battery on } x = r_0\cos\alpha\text{)}. \end{cases} \tag{8}$$

The charge density on the interior of the plates is uniform, with opposite signs on the two plates.

The figure on p. 1 suggests that the charge density on the exterior of the plates is much smaller than that on the interior, and changes sign along the plates. If the potential (5) is approximately valid in the region ($r > r_0$, $|\theta| < \alpha$), then for the case of a curved battery on the surface (r_0, $|\theta| < \alpha$), $E_{\perp,{\rm exterior}} = E_r$ is the same on the exterior and the interior of that surface, $\sigma_{\rm exterior} \approx V_0 \sin\theta/8\pi\sin\alpha = \sigma_{\rm interior}$. For the case of a planar battery on the surface ($x = r_0\cos\alpha$, $|y| < r_0 \sin\alpha$), the exterior surface charge density is similar, *i.e.*, nonzero.

2.3 Magnetic Field and Poynting Vector

The electrical current in the system flows in loops in the x-y plane, so the magnetic field is that of a uniform solenoid with a noncircular cross section. A computation of the magnetic field $\mathbf{B}$ via the Biot-Savart law involves pairs of current elements equidistant in z from the observation point, such that the x- and y-components of the field from each pair cancel, and the field is only in the z-direction. Then, use of the integral form of Ampère's law tells us that the magnetic field is zero in the exterior region, and is the same in the interior region as for a cylindrical solenoid,

$$B_{z,\mathrm{interior}} = \frac{4\pi I}{c} \tag{9}$$

in the interior, where I is the current per unit length in the circuit.[8]

The Poynting vector [28] is nonzero only in the interior region, where the electric field is given by eq. (6) for batteries that obey a suitable boundary condition,

$$\mathbf{S}_{\mathrm{interior}} = \frac{c}{4\pi} \mathbf{E}_{\mathrm{interior}} \times \mathbf{B}_{\mathrm{interior}} = -\frac{V_0 I}{2r_0 \sin\alpha}\,\hat{\mathbf{x}}. \tag{10}$$

The flux of energy from the battery to the resistive plates is uniform, flowing in straight lines parallel to the $-x$ axis as sketched on p. 4. The total power delivered (per unit length) is,

$$P = \int_{-r_0 \sin\alpha}^{r_0 \sin\alpha} |\mathbf{S}_{\mathrm{interior}}|\ dy = V_0 I. \tag{11}$$

That is, all the power, $V_0 I$, consumed by the resistive plates is delivered to them via the energy flow of the electromagnetic field, as described by the Poynting vector.[9,10]

2.4 Momentum

The Poynting vector also has the significance of being c^2 times the density of momentum in the electromagnetic field,[11] so that nonzero field momentum (in the $-x$ direction, and of order $1/c^2$) is associated with the circuit. This is surprising in that the circuit appears to be "at rest", meaning that its center of mass/energy is at rest,[12] and systems "at rest" should have zero total momentum.

[8] See, for example, http://physicspages.com/2013/02/22/solenoid-with-arbitrary-cross-section/, or sec. 10.2.2 of [35].

[9] The resistance R (with dimensions of Ohm-cm) of the plates has not been needed in the preceeding analysis, although it is required to determine the current $I = V_0/2R$ (per unit length) and the magnetic field $B_z = 2\pi V_0/cR$.

[10] An energy-flow velocity can be defined as $\mathbf{v}_{\mathrm{flow}} = \mathbf{S}/u$, where $u = (E^2 + B^2)/8\pi$ is the energy density of the fields. If we take $R = 30$ Ohm-cm $= 1/c$ Ohm-cm, then,

$$v_{\mathrm{flow}} = \frac{16\pi r_0 \sin\alpha}{1 + (16\pi r_0 \sin\alpha)^2} c \le \frac{c}{2}\,. \tag{12}$$

[11] This was first noted by J.J. Thomson in 1891 [36]. See also [37].

[12] Circuits with batteries and resistors involve the transfer of energy/mass from the battery to the resistor, which seems to imply that the center of mass/energy is moving in the direction from the battery to the

The circuit must contain another form of momentum, which was called "hidden momentum" by Shockley [38]. In examples like the present, the mechanical momentum of the moving charges of the electrical current does not sum to zero, but is of order $1/c^2$ (and so can be called a "relativistic" effect) and equal and opposite to the electromagnetic field momentum, such that the total momentum of the system is zero. Details for a circuit with a battery and resistor are discussed in [24]. The idealized case of a circuit with no battery or resistor, in an external electric field, is considered in [39]. General comments on "hidden momentum" are given in [40].

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References

[1] K.T. McDonald, *Charge Density in a Current-Carrying Wire* (Dec. 23, 2010), http://kirkmcd.princeton.edu/examples/wire.pdf

[2] A. Sommerfeld, *Ueber die Fortpflanzung elektrodynamischer Wellen längs eines Drahtes*, Ann. d. Phys. **88**, 233 (1899), http://kirkmcd.princeton.edu/examples/EM/sommerfeld_apc_67_233_99.pdf

[3] G. Mie, *Ein Beispiel für Poyntings Theorem*, Z. Phys. Chemie **34**, 522 (1900), http://kirkmcd.princeton.edu/examples/EM/mie_zpc_34_522_00.pdf

[4] W.S. Franklin, *Poynting's Theorem and the Distribution of Electric Field inside and outside of a Conductor Carrying Electric Current*, Phys. Rev. **13**, 165 (1901), http://kirkmcd.princeton.edu/examples/EM/franklin_pr_13_165_01.pdf

[5] A. Marcus, *The Electric Field Associated with a Steady Current in a Long Cylindrical Conductor*, Am. J. Phys. **9**, 225 (1941), http://kirkmcd.princeton.edu/examples/EM/marcus_ajp_9_225_41.pdf

[6] A. Sommerfeld, *Electrodynamics* (Academic Press, New York, 1952), sec. 17.

[7] O. Jefimenko, *Electricity and Magnetism*, 2nd ed. (Electret Scientific, 1989; 1st ed. 1966), http://kirkmcd.princeton.edu/examples/EM/jefimenko_EM_prob9-6-2.pdf
http://kirkmcd.princeton.edu/examples/EM/jefimenko_EM.pdf

[8] W.G.V. Rosser, *What Makes an Electric Current "Flow"*, Am. J. Phys. **31**, 884 (1963), http://kirkmcd.princeton.edu/examples/EM/rosser_ajp_31_884_63.pdf

[9] B.R. Russell, *Surface Charges on Conductors Carrying Steady Currents*, Am. J. Phys. **36**, 527 (1968), http://kirkmcd.princeton.edu/examples/EM/russell_ajp_36_527_68.pdf

[10] B.R. Russell, *Surface Charges on Conductors Carrying Steady Currents*, Am. J. Phys. **36**, 527 (1968), http://kirkmcd.princeton.edu/examples/EM/russell_ajp_36_527_68.pdf

resistor. However, more detailed consideration of the electrical current as consisting of moving charges with space between them shows that the time average of the "Zitterbewegung" of the center of mass of the moving charges is equal and opposite to the effect on the center of mass/energy due to the flow of energy, such that the center of mass/energy of the entire system is "at rest" on average. See sec. 2.2.2 of [24].

[11] T.N. Sarachman, *The Electric Field Outside a Long Straight Wire Carrying a Steady Current*, Am. J. Phys. **37**, 748 (1969), http://kirkmcd.princeton.edu/examples/EM/sarachman_ajp_37_748_69.pdf

[12] W.G.V. Rosser, *Magnitudes of Surface Charge Distributions Associated with Electric Current Flow*, Am. J. Phys. **38**, 265 (1970), http://kirkmcd.princeton.edu/examples/EM/rosser_ajp_38_265_70.pdf

[13] S. Parker, *Electrostatics and Current Flow*, Am. J. Phys. **38**, 720 (1970), http://kirkmcd.princeton.edu/examples/EM/parker_ajp_38_720_70.pdf

[14] M. Heald, *Electric fields and charges in elementary circuits*, Am. J. Phys. **52**, 522 (1984), http://kirkmcd.princeton.edu/examples/EM/heald_ajp_52_522_84.pdf

[15] C.A. Coombes and H. Laue, *Electric fields and charges distributions associated with steady currents*, Am. J. Phys. **49**, 450 (1981), http://kirkmcd.princeton.edu/examples/EM/coombes_ajp_49_450_81.pdf

[16] R.N. Varney and L.N. Fisher, *Electric fields associated with stationary currents*, Am. J. Phys. **52**, 1097 (1984), http://kirkmcd.princeton.edu/examples/EM/varney_ajp_52_1097_84.pdf

[17] W.R. Moreau *et al.*, *Charge density in circuits*, Am. J. Phys. **53**, 552 (1985), http://kirkmcd.princeton.edu/examples/EM/moreau_ajp_53_552_85.pdf

[18] H.S. Zapolsky, *On electric fields produced by steady currents*, Am. J. Phys. **56**, 1137 (1988), http://kirkmcd.princeton.edu/examples/EM/zapolsky_ajp_56_1137_88.pdf

[19] W.R. Moreau *et al.*, *Charge distributions on DC circuits and Kirchhoff's laws*, Eur. J. Phys. **10**, 286 (1989), http://kirkmcd.princeton.edu/examples/EM/moreau_ejp_10_286_89.pdf

[20] J.M. Aguirregabiria *et al.*, *An example of surface charge distribution or conductors carrying steady currents*, Am. J. Phys. **60**, 138 (1992), http://kirkmcd.princeton.edu/examples/EM/aguirregabiria_ajp_60_138_92.pdf

[21] J.D. Jackson, *Surface charges on circuit wires and resistors play three roles*, Am. J. Phys. **64**, 855 (1996), http://kirkmcd.princeton.edu/examples/EM/jackson_ajp_64_855_96.pdf

[22] J.M. Aguirregabiria *et al.*, *Surface charges and energy flow in a ring rotating in a magnetic field*, Am. J. Phys. **64**, 892 (1996), http://kirkmcd.princeton.edu/examples/EM/aguirregabiria_ajp_64_892_96.pdf

[23] J.A. Hernandes and A.K.T. Assis, *The potential, electric field and surface charges for a resistive long straight strip carrying a steady current*, Am. J. Phys. **71**, 938 (2003), http://kirkmcd.princeton.edu/examples/EM/hernandes_ajp_71_938_03.pdf

[24] K.T. McDonald, *"Hidden" Momentum in a Coaxial Cable* (Mar. 28, 2002), http://kirkmcd.princeton.edu/examples/hidden.pdf

[25] M.K. Harbola, *Energy flow from a battery to other circuit elements: Role of surface charges*, Am. J. Phys. **78**, 1203 (2010),
http://kirkmcd.princeton.edu/examples/EM/harbola_ajp_78_1203_10.pdf

[26] D.V. Redžić, *Charges and fields in a current-carrying wire*, Eur. J. Phys. **33**, 512 (2012),
http://kirkmcd.princeton.edu/examples/EM/redzic_ejp_33_513_12.pdf

[27] R. Folman, *On the possibility of a relativistic correction to the* **E** *and* **B** *fields around a current-carrying wire*, J. Phys. Conf. Ser. **437**, 012013 (2013),
http://kirkmcd.princeton.edu/examples/EM/folman_jpcs_437_012013_13.pdf

[28] J.H. Poynting, *On the Transfer of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **175**, 343 (1884),
http://kirkmcd.princeton.edu/examples/EM/poynting_ptrsl_175_343_84.pdf

[29] P.S. Wedell-Wedellsborg, *Notiz über Poyntings Theorem*, Z. Phys. Chemie **22**, 212 (1897), http://kirkmcd.princeton.edu/examples/EM/wedell-wedellsborg_zpc_22_212_97.pdf
Ueber die Gültigkeit der Maxwellschen Gleichungen, Z. Phys. Chemie **24**, 367 (1897),
http://kirkmcd.princeton.edu/examples/EM/wedell-wedellsborg_zpc_24_367_97.pdf
Antwort an Herrn Scheye, Z. Phys. Chemie **29**, 494 (1899),
Widerung eines sehr allgemeinen und wichtigen Satzes der modernen Elektrizitätslehre, Z. Phys. Chemie **33**, 631 (1900),
http://kirkmcd.princeton.edu/examples/EM/wedell-wedellsborg_zpc_33_631_00.pdf
Über Poyntings Theorem, Z. Phys. Chemie **35**, 604 (1900),
http://kirkmcd.princeton.edu/examples/EM/wedell-wedellsborg_zpc_35_604_00.pdf

[30] D.A. Goldhammer, *Einige Bemerkungen über die von Herrn Wedell-Wedellsborg vermutete Nichtgültigkeit der Maxwellschen Gleichungen für das Innere der Konduktoren*, Z. Phys. Chemie **23**, 686 (1897),
http://kirkmcd.princeton.edu/examples/EM/goldhammer_zpc_23_686_97.pdf

[31] A. Scheye, *Bemerkung zu der Abhandlung des Hernn Wedell-Wedellsborg*, Z. physikal. Chemie **26**, 159 (1898), http://kirkmcd.princeton.edu/examples/EM/scheye_zpc_26_159_98.pdf
Erwiderung auf die "Antwort" des Herrn Wedell-Wedellsborg, Z. physikal. Chemie **32**, 145 (1900), http://kirkmcd.princeton.edu/examples/EM/scheye_zpc_32_145_00.pdf

[32] K.T. McDonald, *Momentum in a DC Circuit* (May 26, 2006),
http://kirkmcd.princeton.edu/examples/loop.pdf

[33] J.A. Stratton, *Electromagnetic Theory* (McGraw-Hill, 1941),
http://kirkmcd.princeton.edu/examples/EM/stratton_electromagnetic_theory.pdf

[34] W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2^{nd} ed. (Addison-Wesley, 1962), http://kirkmcd.princeton.edu/examples/EM/panofsky-phillips.pdf

[35] A. Zangwill, *Modern Electrodynamics* (Cambridge U. Press, 2012),
http://kirkmcd.princeton.edu/examples/EM/zangwill_em_12.pdf

[36] J.J. Thomson, *On the Illustration of the Properties of the Electric Field by Means of Tubes of Electrostatic Induction*, Phil. Mag. **31**, 149 (1891),
http://kirkmcd.princeton.edu/examples/EM/thomson_pm_31_149_91.pdf

[37] K.T. McDonald, *J.J. Thomson and "Hidden" Momentum* (Apr. 30, 2014),
http://kirkmcd.princeton.edu/examples/thomson.pdf

[38] W. Shockley and R.P. James, *"Try Simplest Cases" Discovery of "Hidden Momentum" Forces on "Magnetic Currents,"* Phys. Rev. Lett. **18**, 876 (1967),
http://kirkmcd.princeton.edu/examples/EM/shockley_prl_18_876_67.pdf

[39] K.T. McDonald, *"Hidden" Momentum in a Current Loop* (June 30, 2012),
http://kirkmcd.princeton.edu/examples/penfield.pdf

[40] K.T. McDonald, *On the Definition of "Hidden" Momentum* (July 9, 2012),
http://kirkmcd.princeton.edu/examples/hiddendef.pdf