Potentials for an Electromagnetic Plane Wave

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1 Problem

Discuss the potentials V and A in various gauges for a plane electromagnetic wave in vacuum such as,

$$\mathbf{E} = \cos(kz - \omega t)\,\hat{\mathbf{x}}, \qquad \mathbf{B} = \cos(kz - \omega t)\,\hat{\mathbf{y}}, \tag{1}$$

where $\omega = kc$ and c is the speed of light in vacuum.

2 Solution

2.1 Potentials and Gauge Transformations

Faraday discovered (as later interpreted by Maxwell) that,

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\,, \tag{2}$$

where c is the speed of light in vacuum, which implies that time-dependent magnetic fields $\mathbf{B}$ are associated with additional electric fields beyond those deducible from a scalar potential V. The nonexistence (so far as we know) of magnetic charges (Gilbertian monopoles) implies that,

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{3}$$

and hence that the magnetic field can be related to a vector potential $\mathbf{A}$ by,

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{4}$$

Using eq. (4) in (2), we can write,

$$\boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) = 0, \tag{5}$$

which implies that $\mathbf{E} + (1/c)\partial\mathbf{A}/\partial t$ can be related to a scalar potential V as $-\boldsymbol{\nabla} V$, *i.e.*,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\,. \tag{6}$$

Then, using eq. (6) in the Maxwell equation,

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho \tag{7}$$

leads to,

$$\nabla^2 V + \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\mathbf{A} = -4\pi\rho. \tag{8}$$

Similarly, using eqs. (4) and (6) in the Maxwell equation,

$$\boldsymbol{\nabla}\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}, \tag{9}$$

where $\mathbf{J}$ is the volume density of electrical current, leads to,

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J} + \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c}\frac{\partial V}{\partial t}\right). \tag{10}$$

The differential equations (8) and (10) do not uniquely determine the potentials V and $\mathbf{A}$. As perhaps first clearly noted by Lorentz [1, 2],[1] if V_0, $\mathbf{A}_0$ are valid electromagnetic potentials, then so are,

$$V = V_0 - \frac{1}{c}\frac{\partial\chi}{\partial t}, \qquad \mathbf{A} = \mathbf{A}_0 + \boldsymbol{\nabla}\chi, \tag{11}$$

where χ is an arbitrary scalar function, now called the gauge-transformation function. That is, eqs. (4) and (6) give the same values for the electromagnetic fields $\mathbf{B}$ and $\mathbf{E}$ for either the potentials V, $\mathbf{A}$ or V_0, $\mathbf{A}_0$.

2.2 Gibbs Gauge

Perhaps the simplest potentials for the plane wave (1) are those in the Gibbs gauge [5, 6, 7], in which the gauge condition is that the scalar potential V is zero,

$$V^{(\mathrm{G})} = 0 \qquad \text{(Gibbs gauge)}. \tag{12}$$

For the plane wave (1), the relations that $\mathbf{E} = -(1/c)d\mathbf{A}^{(\mathrm{G})}/dt$ and $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}^{(\mathrm{G})}$ imply that,

$$A^{(\mathrm{G})} = \frac{c}{\omega}\sin(kz - \omega t)\,\hat{\mathbf{x}} + \mathbf{F}(\mathbf{r}), \tag{13}$$

where $\mathbf{F}$ is any vector function of space, but not of time, whose curl is zero. Then,

$$\mathbf{E}^{(\mathrm{G})} = -\boldsymbol{\nabla}V^{(\mathrm{G})} - \frac{1}{c}\frac{\partial\mathbf{A}^{(\mathrm{G})}}{\partial t} = \mathbf{E}, \qquad \mathbf{B}^{(\mathrm{G})} = \boldsymbol{\nabla}\times\mathbf{A}^{(\mathrm{G})} = \mathbf{B}. \tag{14}$$

If the take $\mathbf{F} = 0$, then,

$$\boldsymbol{\nabla}\cdot\mathbf{A}^{(\mathrm{G})} = 0 = -\frac{1}{c}\frac{dV^{(\mathrm{G})}}{dt}. \tag{15}$$

[1]A transformation $\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi$ of the vector potential was discussed by W. Thomson (1850) in sec. 82 of [3], without consideration of the electric field/potential. In sec. 98 of [4], Maxwell noted that if potentials V_0, $\mathbf{A_0}$ do not obey $\boldsymbol{\nabla}\cdot\mathbf{A}_0 = 0$, then the potentials V and $\mathbf{A}$ of eq. (11) [Maxwell's eqs. (74) and(77)] obey $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$ (Coulomb gauge) if $\nabla^2\chi = \boldsymbol{\nabla}\cdot\mathbf{A}_0$, which he thereafter considered to be the proper type of potentials.

2.3 Coulomb Gauge

The Coulomb-gauge condition is that,

$$\boldsymbol{\nabla} \cdot \mathbf{A}^{(\mathrm{C})} = 0 \qquad \text{(Coulomb)}, \tag{16}$$

so the Gibbs-gauge potentials (12)-(13) with $\mathbf{F} = 0$ are also Coulomb-gauge potentials for the plane wave (1).

However, there is an infinite set of Coulomb-gauge potentials that can be obtained from one another via so-called restricted gauge transformations of the form (11) using gauge functions χ that obey $\nabla^2 \chi = 0$. For example, consider,

$$\chi = x \cos \omega t. \tag{17}$$

This gauge function leads to,

$$\mathbf{A}' = \mathbf{A}^{(\mathrm{G})} - \boldsymbol{\nabla}\chi = \frac{\omega}{c} \sin(kz - \omega t)\,\hat{\mathbf{x}} + \cos \omega t\,\hat{\mathbf{x}}, \qquad V' = V^{(\mathrm{G})} + \frac{1}{c}\frac{\partial \chi}{\partial t} = -\frac{\omega}{c} x \sin \omega t, \tag{18}$$

$$\mathbf{E}' = -\boldsymbol{\nabla} V' - \frac{1}{c}\frac{\partial \mathbf{A}'}{\partial t} = \frac{\omega}{c} \sin \omega t\,\hat{\mathbf{x}} + \cos(kz - \omega t)\,\hat{\mathbf{x}} - \frac{\omega}{c} \sin \omega t\,\hat{\mathbf{x}} = cos(kz - \omega t)\,\hat{\mathbf{x}} = \mathbf{E}, \tag{19}$$

$$\mathbf{B}' = \boldsymbol{\nabla} \times \mathbf{A}' = \mathbf{B}. \tag{20}$$

Since $\boldsymbol{\nabla} \cdot \mathbf{A}' = 0$, the potentials (18) are also Coulomb-gauge potentials for the plane wave (1). But, as $V' \neq 0$, the potentials (18) are not in the Gibbs gauge, and the Coulomb- and Gibbs-gauge potentials for a plane electromagnetic wave are distinct in general.

Note also that,

$$\frac{\partial V'}{\partial t} = -\frac{\omega^2}{c} x \cos \omega t. \tag{21}$$

2.4 Lorenz Gauge

The Lorenz-gauge condition [8] is that,

$$\boldsymbol{\nabla} \cdot \mathbf{A}^{(\mathrm{L})} = -\frac{1}{c}\frac{\partial V^{(\mathrm{L})}}{\partial t} \qquad \text{(Lorenz)}, \tag{22}$$

so that the Gibbs-gauge potentials (12)-(13) with $\mathbf{F} = 0$ are also Lorenz-gauge potentials (as well as Coulomb-gauge potentials), but the Coulomb-gauge potentials (18) are not Lorenz-gauge potentials in view of $\boldsymbol{\nabla} \cdot \mathbf{A}' = 0$ and eq. (21).

There is an infinite set of Lorenz-gauge potentials that can be obtained from one another via so-called restricted gauge transformations of the form (11) using gauge functions χ that obey the scalar wave equation,

$$\nabla^2 \chi = \frac{1}{c^2}\frac{\partial^2 \chi}{\partial t^2}\,. \tag{23}$$

For example, consider,

$$\chi = \cos(kz - \omega t). \tag{24}$$

This gauge function leads to,

$$\mathbf{A}'' = \mathbf{A}^{(\mathrm{G})} - \boldsymbol{\nabla}\chi = \frac{\omega}{c}\sin(kz - \omega t)\,\hat{\mathbf{x}} + k\sin(kz - \omega t)\,\hat{\mathbf{z}}, \qquad V'' = V^{(\mathrm{G})} + \frac{1}{c}\frac{\partial\chi}{\partial t} = \frac{\omega}{c}\sin(kz - \omega t), \tag{25}$$

$$\mathbf{E}'' = -\boldsymbol{\nabla}V'' - \frac{1}{c}\frac{\partial\mathbf{A}''}{\partial t} = \frac{\omega}{c}\sin(kz - \omega t)\,\hat{\mathbf{z}} + \cos(kz - \omega t)\,\hat{\mathbf{x}} - \frac{\omega}{c}\sin(kz - \omega t)\,\hat{\mathbf{x}} = \mathbf{E}, \tag{26}$$

$$\mathbf{B}'' = \boldsymbol{\nabla}\times\mathbf{A}'' = \mathbf{B}, \tag{27}$$

$$\boldsymbol{\nabla}\cdot\mathbf{A}'' = k^2\cos(kz - \omega t) = -\frac{1}{c}\frac{\partial V''}{\partial t}\,. \tag{28}$$

Hence, the potentials (18) are also Lorenz-gauge potentials for the plane wave (1).

Since V'' is nonzero the potentials (18) are not in the Gibbs gauge, and as $\boldsymbol{\nabla}\cdot\mathbf{A}''$ is nonzero, they are not Coulomb-gauge potentials.

Thus, in general, the Gibbs-gauge, Coulomb-gauge and Lorenz-gauge potentials for the plane wave (1) are distinct.[2]

This problem was suggested by Vladimir Onoochin.

References

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[2] H.A. Lorentz, *The Theory of Electrons* (Teubner, 1909), Note 5, pp. 238-241, http://kirkmcd.princeton.edu/examples/EM/lorentz_theory_of_electrons_09.pdf

[3] W. Thomson, *A Mathematical Theory of Magnetism*, Phil. Trans. Roy. Soc. London **141**, 269 (1851), http://kirkmcd.princeton.edu/examples/EM/thomson_ptrsl_141_269_51.pdf

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[5] J.W. Gibbs, *Velocity of Propagation of Electrostatic Forces*, Nature **53**, 509 (1896), http://kirkmcd.princeton.edu/examples/EM/gibbs_nature_53_509_96.pdf

[6] K.T. McDonald, *Potentials of a Hertzian Dipole in the Gibbs Gauge* (Aug. 23, 2012), http://kirkmcd.princeton.edu/examples/gibbs.pdf

[7] K.-H. Yang and K.T. McDonald, *Formal Expressions for the Electromagnetic Potentials in Any Gauge* (Feb. 25, 2015), http://kirkmcd.princeton.edu/examples/gauge.pdf

[2]Gibbs [5] supposed that a key aspect of Maxwell's potentials for electromagnetic waves (in the Coulomb gauge) was that the scalar potential V be zero. That is, he implicitly assumed that the gauge conditions $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$ and $V = 0$ were equivalent. The present note illustrates that this is not so, and Gibbs-gauge condition $V = 0$ leads to a gauge different than the Coulomb gauge.

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