Transverse Waves on an Inelastic Vertical String

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1 Problem

What are the frequencies of small transverse oscillations in a vertical plane of an inelastic string of length l and linear mass density λ whose upper point is fixed at a point in a uniform gravitational field of strength g?

Estimate the lowest oscillation frequency via Rayleigh's energy method using, say, a trial waveform $s(y) = l^p - y^p$ for y measured upwards from the lower end of the string, where p is to be optimized.

2 Solution

The equilibrium state of the string is, of course, that it hangs vertically, with its lower end at $y = 0$ and its upper end at $y = l$.

The tension in the string is,

$$T(y) = \lambda g y. \tag{1}$$

The equation of motion for a transverse displacement $s(y, t)$ in a vertical plane of a segment dy of the string is

$$\lambda\, dx\, \ddot{s} = T(y + dy)s'(y + dy) - T(y)s'(y) = \frac{\partial T s'}{\partial y}\, dy = \lambda g \frac{\partial (y s')]}{\partial y}\, dy \tag{2}$$

For oscillations at angular frequency ω of the form $s(y, t) = s(y)e^{i\omega t}$, eq. (2) reduces to,

$$\frac{d(ys')}{dy} + \frac{\omega^2}{g} s = y\frac{d^2 s}{dy^2} + \frac{ds}{dy} + \frac{\omega^2}{g} s = 0. \tag{3}$$

This is a form of Bessel's equation of order zero, as can be seen using the substitution $x = \sqrt{y}$, with which eq. (3) becomes,

$$x^2 \frac{d^2 s}{dx^2} + x\frac{ds}{dx} + \frac{4\omega^2}{g} x^2 s = 0, \tag{4}$$

whose solutions are,

$$s(y) = s_0 J_0(2\omega\sqrt{y/g}). \tag{5}$$

The condition that $s(y = l) = 0$ determine a series of frequencies of small oscillation,

$$2\omega\sqrt{\frac{l}{g}} = 2.405,\ 5.520,\ 8.654, \ldots, \tag{6}$$

or,

$$\omega = 1.202\sqrt{\frac{g}{l}}\,,\ 2.760\sqrt{\frac{g}{l}}\,,\ 4.318\sqrt{\frac{g}{l}}\,,\ldots \tag{7}$$

Rayleigh noted that for a springlike system, $\langle \mathrm{KE}\rangle = \langle \mathrm{PE}\rangle$ (virial theorem), so that a trial waveform with parameter p can be used to estimate the frequency $\omega(p)$ using this constraint. Then the lowest frequency is obtained by minimizing $\omega(p)$ with respect to the parameter p.

We consider the form,

$$s(y,t) = (l^p - y^p)e^{i\omega t}, \tag{8}$$

for which the time-average kinetic energy is,

$$\begin{aligned}
\langle \mathrm{KE}\rangle &= \left\langle \int_0^l \frac{\lambda \dot{s}^2}{2}\,dy \right\rangle = \frac{\lambda\omega^2}{4}\int_0^l (l^p - y^p)^2\,dy = \frac{\lambda\omega^2}{4} l^{2p+1}\left(1 - \frac{2}{p+1} + \frac{1}{2p+1}\right) \\
&= \frac{\lambda\omega^2}{4} l^{2p+1}\frac{2p^2}{(p+1)(2p+1)}\,,
\end{aligned} \tag{9}$$

and the time-average potential energy (= work done in stretching the string) is,

$$\langle \mathrm{PE}\rangle = \left\langle \int_0^l T(\sqrt{1+s'^2}-1)\,dy \right\rangle \approx \left\langle \int_0^l \frac{Ts'^2}{2}\,dy \right\rangle = \frac{\lambda g}{4}\int_0^l y(-py^{p-1})^2\,dy = \frac{\lambda g}{4} l^{2p}\frac{p}{2}\,. \tag{10}$$

Equating the kinetic and potential energies, we have that,

$$\omega^2(p) = \frac{g}{l}\frac{(p+1)(2p+1)}{4p}\,. \tag{11}$$

The minimum frequency occurs for $p = 1/\sqrt{2}$, which implies that its value is,

$$\omega \approx \sqrt{\frac{g}{l}}\sqrt{\frac{1.707\cdot 2.414}{2.828}} = 1.207\sqrt{\frac{g}{l}}\,, \tag{12}$$

which compares well with the "exact" value of $1.202\sqrt{g/l}$.

For additional discussion, see A.B. Western, Demonstration for observing $J_0(x)$ on a resonant rotating vertical chain, Am. J. Phys. **48***, 54 (1980),*
http://kirkmcd.princeton.edu/examples/mechanics/western_ajp_48_54_80.pdf
An early paper on this topic is by J.H. Rohrs, Oscillations of a Suspension Chain, Trans. Camb. Phil. Soc. **9***, Part III, 49 (1851),*
http://kirkmcd.princeton.edu/examples/mechanics/rohrs_tcps_9(3)_49_51.pdf