Transverse Waves on an Inelastic Vertical String

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1 Problem

What are the frequencies of small transverse oscillations in a vertical plane of an inelastic string of length \( l \) and linear mass density \( \lambda \) whose upper point is fixed at a point in a uniform gravitational field of strength \( g \)?

Estimate the lowest oscillation frequency via Rayleigh’s energy method using, say, a trial waveform \( s(y) = l^p - y^p \) for \( y \) measured upwards from the lower end of the string, where \( p \) is to be optimized.

2 Solution

The equilibrium state of the string is, of course, that it hangs vertically, with its lower end at \( y = 0 \) and its upper end at \( y = l \).

The tension in the string is,
\[
T(y) = \lambda gy.
\] (1)

The equation of motion for a transverse displacement \( s(y, t) \) in a vertical plane of a segment \( dy \) of the string is
\[
\lambda \frac{dx}{dy} \ddot{s} = T(y + dy)s'(y + dy) - T(y)s'(y) = \frac{\partial T s'}{\partial y} dy = \lambda g \frac{\partial (ys')}{\partial y} dy
\] (2)

For oscillations at angular frequency \( \omega \) of the form \( s(y, t) = s(y) e^{i\omega t} \), eq. (2) reduces to,
\[
\frac{d(ys')}{dy} + \frac{\omega^2}{g} s = y \frac{d^2 s}{dy^2} + \frac{ds}{dy} + \frac{\omega^2}{g} s = 0.
\] (3)

This is a form of Bessel’s equation of order zero, as can be seen using the substitution \( x = \sqrt{\frac{y}{g}} \), with which eq. (3) becomes,
\[
x^2 \frac{d^2 s}{dx^2} + x \frac{ds}{dx} + \frac{4\omega^2}{g} x^2 s = 0,
\] (4)

whose solutions are,
\[
s(y) = s_0 J_0(2\omega \sqrt{y/g}).
\] (5)

The condition that \( s(y = l) = 0 \) determine a series of frequencies of small oscillation,
\[
2\omega \sqrt{\frac{l}{g}} = 2.405, \ 5.520, \ 8.654, \ldots,
\] (6)
or,

$$\omega = 1.202 \sqrt{\frac{g}{l}}, \quad 2.760 \sqrt{\frac{g}{l}}, \quad 4.318 \sqrt{\frac{g}{l}}, \ldots$$  \hspace{1cm} (7)

Rayleigh noted that for a springlike system, $\langle \text{KE} \rangle = \langle \text{PE} \rangle$ (virial theorem), so that a trial waveform with parameter $\omega(p)$ can be used to estimate the frequency $\omega(p)$ using this constraint. Then the lowest frequency is obtained by minimizing $\omega(p)$ with respect to the parameter $p$. We consider the form,

$$s(y, t) = (lp - y^p)e^{i\omega t},$$  \hspace{1cm} (8)

for which the time-average kinetic energy is,

$$\langle \text{KE} \rangle = \frac{\lambda}{4} \int_0^l \frac{\lambda s'^2}{2} dy = \frac{\lambda \omega^2}{4} l^{2p+1} \left( 1 - \frac{2}{p+1} + \frac{1}{2p+1} \right)$$

$$= \frac{\lambda \omega^2}{4} l^{2p+1} \frac{2p^2}{(p+1)(2p+1)},$$  \hspace{1cm} (9)

and the time-average potential energy (= work done in stretching the string) is,

$$\langle \text{PE} \rangle = \frac{\lambda}{4} \int_0^l T\left(\sqrt{1 + s'^2} - 1\right) dy \approx \frac{\lambda}{4} \int_0^l Ts'^2 dy = \lambda \frac{g^2}{4} l^{2p+1} \frac{1}{2} (p+1).$$

Equating the kinetic and potential energies, we have that,

$$\omega^2(p) = \frac{g}{l} \frac{(p+1)(2p+1)}{4p}.$$  \hspace{1cm} (10)

The minimum frequency occurs for $p = 1/\sqrt{2}$, which implies that its value is,

$$\omega \approx \sqrt{\frac{g}{l}} \sqrt{\frac{1.707 \cdot 2.414}{2.828}} = 1.207 \sqrt{\frac{g}{l}},$$  \hspace{1cm} (11)

which compares well with the “exact” value of $1.202 \sqrt{g/l}$.

For additional discussion, see A.B. Western, Demonstration for observing $J_0(x)$ on a resonant rotating vertical chain, Am. J. Phys. 48, 54 (1980),

http://kirkmcd.princeton.edu/examples/mechanics/western_ajp_48_54_80.pdf

An early paper on this topic is by J.H. Rohrs, Oscillations of a Suspension Chain, Trans. Camb. Phil. Soc. 9, Part III, 49 (1851),

http://kirkmcd.princeton.edu/examples/mechanics/rohrs_tcps_9(3)_49_51.pdf