Direct, analytic solution for the electromagnetic vector potential in any gauge

Kuo-Ho Yang
Department of Engineering and Physics, St. Ambrose University, Davenport, IA 52803
E-mail: yangkuoho@sau.edu
Robert D. Nevels
Department of Electrical and Computer Engineering, Texas A&M University, College Station, TX 77843
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Abstract

We propose a new method for deriving an analytic solution for the electromagnetic vector potential in any gauge directly from Maxwell's equations for potentials..

Keywords: classical electrodynamics, Maxwell's equations, vector potential

1 Introduction

To understand gauge transformations or gauge invariance in classical electrodynamics, it is essential that we have detailed knowledge of the vector and the scalar potentials. In general, the potentials are solved from Maxwell's equations for potentials. But, Maxwell's equations for potentials contain a gauge ambiguity. If two sets of potentials are related by a gauge transformation, then both sets are solutions of the equations (*e.g.*, Refs. [1, 2]).

Thus, to solve for a *unique* set of potentials, we add a constraint involving the vector or the scalar potential or both. Such a constraint is called a gauge condition. Then, we solve for the potentials from the resulting equations after the gauge condition is applied. But, each gauge condition generates its own equations for the potentials and hence its own mathematical challenges. For example, both the scalar and the vector potentials in the Lorenz gauge can be solved simply. But, the situation is not as straightforward for the Coulomb gauge. Although the Coulomb-gauge scalar potential is easy to solve, there are no general analytic solutions for the vector potential in Ref. [2]. It was only recently that general analytic solutions were obtained the vector potential in the Coulomb gauge [3–14].

In this paper, we propose a new method for deriving an analytic solution for the vector potential in any gauge for an arbitrary charge-current distribution. The vector potential is solved *directly* from Maxwell's equations for potentials *without* using a gauge condition. Because no gauge condition is used, the solution is universally valid for any gauge. We show that the fields generated by our potentials in any gauge are gauge invariant and *always* propagate with speed c from physical charge and current densities. In the Appendix, we solve the vector potential using the Fourier transforms to make the mathematics easier to grasp.

2 Direct, analytic solution for the vector potential in any gauge from Maxwell's equations for potentials

We consider localized charge and current densities, $\rho(\mathbf{r},t)$ and $\mathbf{J}(\mathbf{r},t)$, which are turned on at t_0. The electric field $\mathbf{E}$, the magnetic field $\mathbf{B}$, and Maxwell's equations for potentials $\mathbf{A}$ and Φ are related by (in Gaussian units):

$$\mathbf{E}(\mathbf{r},t) = -\boldsymbol{\nabla}\Phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}, \qquad \mathbf{B}(\mathbf{r},t) = \boldsymbol{\nabla}\times\mathbf{A}(\mathbf{r},t), \tag{1}$$

$$\boldsymbol{\nabla}^2\Phi(\mathbf{r},t) + \frac{1}{c}\frac{\partial}{\partial t}[\boldsymbol{\nabla}\cdot\mathbf{A}(\mathbf{r},t)] = -4\pi\rho(\mathbf{r},t), \tag{2}$$

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}(\mathbf{r},t) = -\frac{4\pi}{c}\mathbf{J}(\mathbf{r},t) + \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A}(\mathbf{r},t) + \frac{1}{c}\frac{\partial\Phi(\mathbf{r},t)}{\partial t}\right). \tag{3}$$

For simplicity, we assume that there are no boundary surfaces and that all potentials are subjected to the initial conditions, $\mathbf{A}(\mathbf{r},t) = \Phi(\mathbf{r},t) = 0$ for all $\mathbf{r}$ at $t \leq t_0$, and the boundary conditions, $\mathbf{A}(\mathbf{r},t) = \Phi(\mathbf{r},t) = 0$ for all t at $|\mathbf{r}| \to \infty$.

To solve for the vector potential $\mathbf{A}$ from (2) and (3), we write $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$ where $\mathbf{A}_1$ and $\mathbf{A}_2$ are solutions of their respective equations:

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}_1 = -\frac{4\pi}{c}\mathbf{J}, \tag{4}$$

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}_2 = \boldsymbol{\nabla}\left[\boldsymbol{\nabla}\cdot(\mathbf{A}_1+\mathbf{A}_2) + \frac{1}{c}\frac{\partial\Phi}{\partial t}\right]. \tag{5}$$

From (4), it is clear that the solution for $\mathbf{A}_1$ is the vector potential $\mathbf{A}^{(L)}$ in the Lorenz gauge:

$$\mathbf{A}_1(\mathbf{r},t) = \mathbf{A}^{(L)}(\mathbf{r},t) = \frac{1}{c}\int G(\mathbf{r},t|c|\mathbf{r}',t')\mathbf{J}(\mathbf{r}',t')d^3r'dt', \tag{6}$$

$$G(\mathbf{r},t|c|\mathbf{r}',t') = \frac{\delta\left(t - \frac{|\mathbf{r}-\mathbf{r}'|}{c} - t'\right)}{|\mathbf{r}-\mathbf{r}'|}, \tag{7}$$

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}, t|c|\mathbf{r}', t') = -4\pi\delta(\mathbf{r} - \mathbf{r}')\delta(t - t'). \tag{8}$$

To solve for $\mathbf{A}_2$, we differentiate both sides of (5) with respect to t and use (2) to eliminate $(\partial/\partial t)[\boldsymbol{\nabla} \cdot (\mathbf{A}_1 + \mathbf{A}_2)]$ to get,

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial \mathbf{A}_2}{\partial t}\right) = -4\pi c\boldsymbol{\nabla}\rho - \left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)(c\boldsymbol{\nabla}\Phi). \tag{9}$$

This equation can be solved in terms of the scalar potential $\Phi^{(L)}$ in the Lorenz gauge:

$$\mathbf{A}_2(\mathbf{r}, t) = c\boldsymbol{\nabla}\int_{t_0}^{t}\left[\Phi^{(L)}(\mathbf{r}, \tau) - \Phi(\mathbf{r}, \tau)\right] d\tau, \tag{10}$$

$$\Phi^{(L)}(\mathbf{r}, t) = \int G(\mathbf{r}, t|c|\mathbf{r}', t')\rho(\mathbf{r}', t')d^3r'dt'. \tag{11}$$

Thus, the full expression for the vector potential is:

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}^{(L)}(\mathbf{r}, t) + c\boldsymbol{\nabla}\int_{t_0}^{t}\left[\Phi^{(L)}(\mathbf{r}, \tau) - \Phi(\mathbf{r}, \tau)\right] d\tau. \tag{12}$$

This vector potential clearly satisfies (2) for any Φ:

$$\begin{aligned}\boldsymbol{\nabla}^2\Phi + \frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \mathbf{A}) &= \boldsymbol{\nabla}^2\Phi + \left[\frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \mathbf{A}^{(L)}) + (\boldsymbol{\nabla}^2\Phi^{(L)} - \boldsymbol{\nabla}^2\Phi)\right] \\ &= \boldsymbol{\nabla}^2\Phi^{(L)} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi^{(L)} = -4\pi\rho. \end{aligned} \tag{13}$$

Because (12) is derived without using a gauge condition, it is valid for any gauge.

In the Appendix, we solve the vector potential using the method of Fourier transforms to make the mathematics more transparent.

3 Gauge transformations of potentials, gauge invariance of fields, and potentials in the velocity gauge

In this section, we show that our solution of the vector potential in (12) for any gauge has two important properties that any two sets of potentials are related by a gauge transformation, and that the electric and the magnetic fields generated by the potentials are gauge invariant.

Let us consider two set of potentials (Φ, $\mathbf{A}$) in (12) and (Φ', $\mathbf{A}'$) listed below:

$$\mathbf{A}'(\mathbf{r}, t) = \mathbf{A}^{(L)}(\mathbf{r}, t) + c\boldsymbol{\nabla}\int_{t_0}^{t}\left[\Phi^{(L)}(\mathbf{r}, \tau) - \Phi'(\mathbf{r}, \tau)\right] d\tau. \tag{14}$$

Next, we define a gauge function $\chi(\mathbf{r},t)$ by

$$\chi(\mathbf{r},t) = c\int_{t_0}^{t}\big[\Phi(\mathbf{r},\tau) - \Phi'(\mathbf{r},\tau)\big]\,d\tau. \tag{15}$$

Then these two sets of potentials are related by a gauge transformation via the gauge function χ:

$$\mathbf{A}' = \mathbf{A} + c\boldsymbol{\nabla}\int_{t_0}^{t}\big[\Phi(\mathbf{r},\tau) - \Phi'(\mathbf{r},\tau)\big]\,d\tau = \mathbf{A} + \boldsymbol{\nabla}\chi, \tag{16}$$

$$\Phi' = \Phi + \frac{1}{c}\frac{\partial}{\partial t}\left(c\int_{t_0}^{t}\big[\Phi'(\mathbf{r},\tau) - \Phi(\mathbf{r},\tau)\big]\,d\tau\right) = \Phi - \frac{1}{c}\frac{\partial\chi}{\partial t}. \tag{17}$$

Because of the above two relationships, the electric and the magnetic fields are gauge invariant:

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi' - \frac{1}{c}\frac{\partial\mathbf{A}'}{\partial t} = -\boldsymbol{\nabla}\left(\Phi - \frac{1}{c}\frac{\partial\chi}{\partial t}\right) - \frac{1}{c}\frac{\partial(\mathbf{A}+\boldsymbol{\nabla}\chi)}{\partial t} = -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}, \tag{18}$$

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}' = \boldsymbol{\nabla}\times(\mathbf{A}+\boldsymbol{\nabla}\chi) = \boldsymbol{\nabla}\times\mathbf{A}. \tag{19}$$

It is important to note that the potentials $(\Phi, \mathbf{A})$ in (12) is just one gauge transformation away from the Lorenz-gauge potentials $(\Phi^{(L)}, \mathbf{A}^{(L)})$ via the gauge function Λ defined below:

$$\mathbf{A} = \mathbf{A}^{(L)} + \boldsymbol{\nabla}\Lambda, \qquad \Lambda(\mathbf{r},t) = c\int_{t_0}^{t}\Big[\Phi^{(L)}(\mathbf{r},\tau) - \Phi(\mathbf{r},\tau)\Big]\,d\tau. \tag{20}$$

This gauge-transformation procedure was first used by Jackson to derive the vector potentials in many gauges of interest [3]. Thus, the results of Jackson's method totally agree with our universal solution for the vector potential in (12). See also Refs. [12, 14] for using this procedure in their investigations.

Let us now apply (12) to the velocity gauge with a *real* parameter $v \neq 0$ (the v-gauge). In this gauge, the gauge condition and the equation for the scalar potential are, *e.g.*, Refs. [3–6]:

$$\boldsymbol{\nabla}\cdot\mathbf{A}^{(v)} + \frac{c}{v^2}\frac{\partial\Phi^{(v)}}{\partial t} = 0, \tag{21}$$

$$\boldsymbol{\nabla}^2\Phi^{(v)} - \frac{1}{v^2}\frac{\partial^2\Phi^{(v)}}{\partial t^2} = -4\pi\rho. \tag{22}$$

The solution for the scalar potential is,

$$\Phi^{(v)}(\mathbf{r},t) = \int G(\mathbf{r},t|v|\mathbf{r}',t')\rho(\mathbf{r}',t')d^3r'dt', \tag{23}$$

where the v-propagating Green function $G(\mathbf{r},t|v|\mathbf{r}',t')$ is obtained from (7)-(8) by substituting v for c. According to (12), the vector potential has the form:

$$\mathbf{A}^{(v)}(\mathbf{r},t) = \mathbf{A}^{(L)}(\mathbf{r},t) + c\boldsymbol{\nabla}\int_{t_0}^{t}\left[\Phi^{(L)}(\mathbf{r},\tau) - \Phi^{(v)}(\mathbf{r},\tau)\right]d\tau. \tag{24}$$

This expression for the v-gauge vector potential was first derived by Yang [4, 6] using the arguments that the electric and the magnetic fields are gauge invariant and propagate with speed c from physical charge and current densities. (In Ref. [4], the velocity gauge was called the α-Lorenz gauge, with $v = \alpha c$.)

Later, Jackson [3] used the gauge-transformation procedure in (20) to derive the v-gauge vector potential, producing the same result as in (24):

$$\mathbf{A}^{(\text{new})} = \mathbf{A}^{(L)} + \boldsymbol{\nabla}\Lambda^{(v)}, \qquad \Lambda^{(v)}(\mathbf{r},t) = c\int_{t_0}^{t}\left[\Phi^{(L)}(\mathbf{r},\tau) - \Phi^{(v)}(\mathbf{r},\tau)\right]d\tau. \tag{25}$$

When the parameter v in the velocity gauge is *imaginary* in the form of $v = \pm i\nu$ with any *real* value of $\nu \neq 0$, the gauge is called the generalized ν-Kirchhoff gauge [11]. The gauge condition and the equation for the scalar potential are:

$$\boldsymbol{\nabla}\cdot\mathbf{A}^{(\nu\text{K})} - \frac{c}{\nu^2}\frac{\partial\Phi^{(\nu\text{K})}}{\partial t} = 0, \tag{26}$$

$$\boldsymbol{\nabla}^2\Phi^{(\nu\text{K})} + \frac{1}{\nu^2}\frac{\partial^2\Phi^{(\nu\text{K})}}{\partial t^2} = -4\pi\rho. \tag{27}$$

When $\nu = c$, the ν-Kirchhoff gauge reduces to the original Kirchhoff gauge investigated extensively by Heras [15]. It is obvious that (26)-(27) are a generalization of Heras's idea of extending v in the velocity gauge to include imaginary values. As a consequence, the ν-Kirchhoff scalar potential can formally be expressed as the velocity-gauge scalar potential with an imaginary propagation speed $v = \pm i\nu$.

We use a simple example to see what the potential $\Phi^{(\nu\text{K})}(\mathbf{r},t)$ looks like. We assume a single-frequency charge density of the form:

$$\rho(\mathbf{r},t) = \rho_+(\mathbf{r})e^{i\omega t} + \rho_-(\mathbf{r})e^{-i\omega t}, \qquad \rho_-(\mathbf{r}) = [\rho_+(\mathbf{r})]^*, \tag{28}$$

where * denotes the complex conjugate. We assume that the potential has the same time-dependence:

$$\Phi^{(\nu\text{K})}(\mathbf{r},t) = \Phi_+^{(\nu\text{K})}(\mathbf{r})e^{i\omega t} + \Phi_-^{(\nu\text{K})}(\mathbf{r})e^{-i\omega t}, \qquad \Phi_-^{(\nu\text{K})}(\mathbf{r}) = [\Phi_+^{(\nu\text{K})}(\mathbf{r})]^*. \tag{29}$$

If we use (28)-(29) in (27), we have

$$\left(\boldsymbol{\nabla}^2 - \frac{\omega^2}{\nu^2}\right)\Phi_\pm^{(\nu\text{K})}(\mathbf{r}) = -4\pi\rho_\pm(\mathbf{r}). \tag{30}$$

Hence, the solution for $\Phi^{(\nu\mathrm{K})}$ that goes to zero at $|\mathbf{r}| \to \infty$ is:

$$\Phi^{(\nu\mathrm{K})}(\mathbf{r}, t) = \int \frac{e^{-|\omega/\nu|R}}{R} \left[\rho_+(\mathbf{r}')e^{i\omega t} + \rho_-(\mathbf{r}')e^{-i\omega t}\right] d^3r', \tag{31}$$

where $R = |\mathbf{r} - \mathbf{r}'|$. When both $\omega, \nu \geq 0$, the exponent of the positive-frequency component is: $i\omega t - |\omega/\nu|R = i\omega[t - R/(i\nu)]$, exhibiting an imaginary propagation speed of $i\nu$, first suggested by Heras [15]. (The negative-frequency component is just the complex conjugate of the positive-frequency component.)

4 Electromagnetic fields and their propagation in space, and potentials in the Poincaré gauge

In this section, we examine the propagation of the fields generated by our potentials. The electric and the magnetic fields $\mathbf{E}$ and $\mathbf{B}$ generated by the potentials in an arbitrary gauge, Φ and $\mathbf{A}$ in (12), are:

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\boldsymbol{\nabla}\Phi - \left[\frac{1}{c}\frac{\partial \mathbf{A}^{(L)}}{\partial t} + \boldsymbol{\nabla}\Big(\Phi^{(L)} - \Phi\Big)\right] = -\boldsymbol{\nabla}\Phi^{(L)} - \frac{1}{c}\frac{\partial \mathbf{A}^{(L)}}{\partial t}, \tag{32}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \left\{\mathbf{A}^{(L)}(\mathbf{r}, t) + c\boldsymbol{\nabla}\int_{t_0}^{t} \left[\Phi^{(L)}(\mathbf{r}, \tau) - \Phi(\mathbf{r}, \tau)\right] d\tau\right\} = \boldsymbol{\nabla} \times \mathbf{A}^{(L)}. \tag{33}$$

Because the potentials in the Lorenz gauge always propagate with speed c from the physical charge and current densities ρ and $\mathbf{J}$ according to (11) and (6), the results in (32)-(33) indicate that the fields *always* propagate with the speed c from the charge and current densities ρ and $\mathbf{J}$. We note that this statement is true for *any* scalar potential Φ.

From (32)-(33), two sets of potentials stand out: the potentials ($\Phi^{(L)}$, $\mathbf{A}^{(L)}$) in the Lorenz gauge and the potentials ($\Phi^{(G)}$, $\mathbf{A}^{(G)}$) in the Gibbs gauge [16, 17] (also known as the Hamilton or temporal gauge [1]):

$$\Phi^{(G)} = 0, \qquad \mathbf{A}^{(G)}(\mathbf{r}, t) = \mathbf{A}^{(L)}(\mathbf{r}, t) + c\boldsymbol{\nabla}\int_{t_0}^{t} \Phi^{(L)}(\mathbf{r}, \tau)d\tau = -c\int_{t_0}^{t} \mathbf{E}(\mathbf{r}, \tau)d\tau, \tag{34}$$

$$-\frac{1}{c}\frac{\partial \mathbf{A}^{(G)}}{\partial t} = \mathbf{E}, \qquad \boldsymbol{\nabla} \times \mathbf{A}^{(G)} = -c\int_{t_0}^{t} \boldsymbol{\nabla} \times \mathbf{E}(\mathbf{r}, \tau)d\tau = \int_{t_0}^{t} \frac{\partial \mathbf{B}(\mathbf{r}, \tau)}{\partial \tau}d\tau = \mathbf{B}. \tag{35}$$

We note that $\mathbf{A}^{(G)}$ is just the time-integration of the local, gauge-invariant electric *field*. The vector potential in (12) also can be expressed as:

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}^{(G)}(\mathbf{r}, t) - c\boldsymbol{\nabla}\int_{t_0}^{t} \Phi(\mathbf{r}, \tau)d\tau = -c\int_{t_0}^{t} \left[\mathbf{E}(\mathbf{r}, \tau) + \boldsymbol{\nabla}\Phi(\mathbf{r}, \tau)\right] d\tau. \tag{36}$$

This expression for the vector potential is most suitable for discussions of the potentials in the Poincaré gauge next.

The Poincaré or multipolar gauge (*e.g.*, [3, 18]) is defined by the gauge condition, $\mathbf{r} \cdot \mathbf{A}^{(P)}(\mathbf{r}, t) = 0$, which is cast in the following form for the discussion of this gauge:

$$\int_0^1 \mathbf{r} \cdot \mathbf{A}^{(P)}(u\mathbf{r}, t) du = 0. \tag{37}$$

The mathematics (but not the physical quantities) used here from (37) to (42) closely follows the mathematics in eqs. (9.1)-(9.9) of Ref. [3]. From (36), (37) takes the form for all t,

$$\int_0^1 \left[\mathbf{r} \cdot \mathbf{E}(u\mathbf{r}, t) + \mathbf{r} \cdot \boldsymbol{\nabla}_{u\mathbf{r}} \Phi^{(P)}(u\mathbf{r}, t) \right] du = 0. \tag{38}$$

The integration over u can be done for the scalar potential as follows. We use the spherical coordinates $\mathbf{r} = (r, \theta, \phi)$ to get

$$\int_0^1 \mathbf{r} \cdot \boldsymbol{\nabla}_{u\mathbf{r}} \Phi^{(P)}(u\mathbf{r}, t) du = \int_0^1 r \frac{\partial \Phi^{(P)}(ur, \theta, \phi, t)}{\partial (ur)} du = \Phi^{(P)}(\mathbf{r}, t) - \Phi^{(P)}(\mathbf{0}, t). \tag{39}$$

Thus, the scalar potential is,

$$\Phi^{(P)}(\mathbf{r}, t) = -\int_0^1 \mathbf{r} \cdot \mathbf{E}(u\mathbf{r}, t) du + \Phi^{(P)}(\mathbf{0}, t). \tag{40}$$

We then take gradient of the scalar potential to get (note: $\boldsymbol{\nabla} = \boldsymbol{\nabla}_{\mathbf{r}}$):

$$-\boldsymbol{\nabla} \Phi^{(P)}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) - \frac{1}{c} \frac{\partial}{\partial t} \int_0^1 \mathbf{r} \times u\mathbf{B}(u\mathbf{r}, t) du. \tag{41}$$

We now use (41) in (36) to get the vector potential,

$$\mathbf{A}^{(P)}(\mathbf{r}, t) = -\int_{t_0}^t d\tau \left(\frac{\partial}{\partial \tau} \int_0^1 \mathbf{r} \times u\mathbf{B}(u\mathbf{r}, \tau) du \right) = -\int_0^1 \mathbf{r} \times u\mathbf{B}(u\mathbf{r}, t) du. \tag{42}$$

The above discussion shows that our universal vector potential (36) or (12) also works for the Poincaré or multipolar gauge.

5 Conclusions

In conclusion, we have derived an analytic solution for the vector potential universally valid for any gauge. This is done by solving the vector potential directly from Maxwell's equations for potentials without using a gauge condition. Of course, a gauge condition is still needed to solve for a particular scalar potential. But as soon as the scalar potential is solved, the vector potential in that gauge is completely determined by using the scalar potential in (12) or (36).

A Appendix: Solution of the vector potential in any gauge by Fourier transforms

In this Appendix, we show that the method of Fourier transforms offers a simpler way of obtaining the solution of the vector potential. Define the Fourier component $\tilde{\mathbf{A}}(\mathbf{k},\omega)$ of the vector potential $\mathbf{A}(\mathbf{r},t)$ by:

$$\tilde{\mathbf{A}}(\mathbf{k},\omega) = \int e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}\mathbf{A}(\mathbf{r},t)d^3r dt, \tag{A.1}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{(2\pi)^4}\int e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\tilde{\mathbf{A}}(\mathbf{k},\omega)d^3k d\omega, \tag{A.2}$$

and similarly for the scalar potential Φ, *etc.* Thus, equations (2)-(3) become

$$\mathbf{k}^2\tilde{\Phi} - \frac{\omega}{c}\mathbf{k}\cdot\tilde{\mathbf{A}} = 4\pi\tilde{\rho}, \tag{A.3}$$

$$\left(\mathbf{k}^2 - \frac{\omega^2}{c^2}\right)\tilde{\mathbf{A}} = \frac{4\pi}{c}\tilde{\mathbf{J}} + \mathbf{k}\left(\mathbf{k}\cdot\tilde{\mathbf{A}} - \frac{\omega}{c}\tilde{\Phi}\right). \tag{A.4}$$

To solve for $\tilde{\mathbf{A}}$, we first solve for $\mathbf{k}\cdot\tilde{\mathbf{A}}$ from (A.3) to get

$$\mathbf{k}\cdot\tilde{\mathbf{A}} = \frac{c}{\omega}\left(\mathbf{k}^2\tilde{\Phi} - 4\pi\tilde{\rho}\right). \tag{A.5}$$

We then use (A.5) in (A.4) to have

$$\left(\mathbf{k}^2 - \frac{\omega^2}{c^2}\right)\tilde{\mathbf{A}} = \frac{4\pi}{c}\tilde{\mathbf{J}} + \mathbf{k}\left[\frac{c}{\omega}\left(\mathbf{k}^2\tilde{\Phi} - 4\pi\tilde{\rho}\right) - \frac{\omega}{c}\tilde{\Phi}\right] = \frac{4\pi}{c}\tilde{\mathbf{J}} - \frac{4\pi c}{\omega}\mathbf{k}\tilde{\rho} + \mathbf{k}\frac{c}{\omega}\left(\mathbf{k}^2 - \frac{\omega^2}{c^2}\right)\tilde{\Phi}. \tag{A.6}$$

The solution for $\tilde{\mathbf{A}}$ is:

$$\tilde{\mathbf{A}} = \frac{4\pi}{\mathbf{k}^2 - \omega^2/c^2}\left(\frac{\mathbf{J}}{c} - c\frac{\mathbf{k}}{\omega}\tilde{\rho}\right) + c\frac{\mathbf{k}}{\omega}\tilde{\Phi} = \tilde{\mathbf{A}}^{(L)} + c\frac{i\mathbf{k}}{-i\omega}(\tilde{\Phi}^{(L)} - \tilde{\Phi}), \tag{A.7}$$

which is exactly (12).

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