“Hidden” Momentum in an Oscillating Tube of Water?

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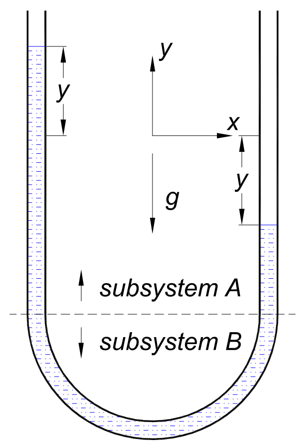
1 Problem

Discuss the motion of “frictionless” water in a vertical U-tube when the tube is free to move horizontally but not vertically.[1] The water levels in the two arms of the tube are initially unequal. For simplicity, neglect the mass of the tube compared to that of the water.



Designate the water in the vertical arms of the U-tube as subsystem A, and the water in the half-circle arc of the tube as subsystem B. Discuss the horizontal components of the velocities of the center of masses, and of the momentum, of the two subsystems (whose masses are equal). Do the subsystems contain hidden momentum, $\mathbf{P}_{\rm hidden}$, defined by,

$$\mathbf{P}_{\rm hidden} \equiv \mathbf{P} - M\mathbf{v}_{\rm cm} - \oint_{\rm boundary} (\mathbf{x} - \mathbf{x}_{\rm cm})\,(\mathbf{p} - \rho\mathbf{v}_b) \cdot d\mathbf{Area}, \tag{1}$$

where $\mathbf{p}$ is the momentum density of the subsystem, ρ is its mass density, $M = \int \rho\, d{\rm Vol}$, and $\mathbf{v}_b$ is the velocity (field) of its boundary?[2]

2 Solution

Taking the total mass of the water to be m and the length of the water in the tube to be l, the equation of motion for the water is,

$$m\ddot{y} = -\frac{2ymg}{l}\,, \tag{2}$$

[1]The tube is also constrained not to rotate.
[2]The definition (1) was inspired by discussions with Daniel Vanzella [1]. See also [2].

when the level of the water in, say, the left arm of the U-tube is height y above the equilibrium value $y = 0$. Thus, the water level oscillates vertically according to,

$$y(t) = y_0 \cos \omega t, \qquad \text{where} \qquad \omega = \sqrt{\frac{2g}{l}}\,. \tag{3}$$

If the system is free to move horizontally, then its center of mass must have constant horizontal velocity, which we take to be zero. The masses of water in the left and right arms of the U-tube oscillate, and hence the system must oscillate horizontally to keep the center of mass at rest.

Assuming that the masses of subsystems A and B are equal (and each equal to $m/2$), the separation between the centers of the two arms of the U-tube is $d = l/\pi$. The center of mass of the system, relative to the center of the U-tube, is,

$$x^\star_{\rm cm}(t) = \frac{[2y(t)/l]\, m\, [-d/2]}{m} = -\frac{y(t)}{\pi}\,, \qquad v^\star_{{\rm cm},x}(t) = -\frac{\dot{y}(t)}{\pi} = \frac{\omega y_0}{\pi} \sin \omega t. \tag{4}$$

In the lab frame, the center of mass of the system is always at $x = 0$, which implies that the center, x_0, of the U-tube in the lab frame oscillates according to,

$$x_0(t) = \frac{y(t)}{\pi} = \frac{y_0}{\pi} \cos \omega t. \tag{5}$$

In the frame of the U-tube, subsystem A has zero horizontal momentum, $P^\star_{{\rm A},x} = 0$, but it center of mass oscillates according to,

$$x^\star_{\rm cm,A}(t) = \frac{[2y(t)/l]\, m\, [-d/2]}{m/2} = -\frac{2y(t)}{\pi} = -\frac{2y_0}{\pi} \cos \omega t, \tag{6}$$

so the velocity of the center of mass of subsystem A is,

$$v^\star_{{\rm cm,A},x}(t) = \frac{2\omega y_0}{\pi} \sin \omega t. \tag{7}$$

This is a peculiar situation in that the subsystem has no x-momentum, but its center of mass velocity has an x-component. Perhaps this subsystem has hidden momentum, as defined by eq. (1).

In the present example, the momentum density has magnitude $p^\star_x = mv/AL = m\omega y_0 \sin(\omega t)/AL$, where A is the cross-sectional area of the tube of water. Then, for subsystem A in the frame of the U-tube, the integral over the bounding surface (which is at rest) has x-component,

$$\oint_{\rm boundary,A} (\mathbf{x}^\star - \mathbf{x}^\star_{\rm cm})\, \mathbf{p}^\star \cdot d\mathbf{Area}^\star = \frac{mvd}{L} = -\frac{m\omega y_0 \sin \omega t}{\pi}\,, \tag{8}$$

and so, according to definition eq. (1),

$$P^\star_{{\rm A,hidden},x} = P^\star_{{\rm A},x} - \frac{m}{2} v^\star_{{\rm cm,A},x} + \frac{m\omega y_0 \sin \omega t}{\pi} = 0. \tag{9}$$

Meanwhile, subsystem B is at rest relative to the U-tube, so that $\mathbf{v}^\star_{\rm cm,B} = 0$. However, the flow of water in the half circle of the U-tube is associated with horizontal momentum,

$$P^\star_{{\rm B},x} = \int_0^\pi \frac{m/2\, d\phi}{\pi} (-\dot{y} \sin\phi) = -m \frac{\omega y_0}{\pi} \sin\omega t. \tag{10}$$

This is also a peculiar situation in that the subsystem possesses momentum but its center of mass is not moving. However, according to the definition (1), subsystem B has hidden momentum (in the frame of the U-tube),

$$P^\star_{{\rm B,hidden},x} = P^\star_{{\rm B},x} - \oint_{\rm boundary,B} (\mathbf{x}^\star - \mathbf{x}^\star_{\rm cm})\, \mathbf{p}^\star \cdot d\mathbf{Area}^\star = 0, \tag{11}$$

as the boundary integral for subsystem B is the negative of that for subsystem A.

The total hidden momentum of the system in the frame of the U-tube is,

$$P^\star_{{\rm hidden},x} = P^\star_{{\rm A,hidden},x} + P^\star_{{\rm B,hidden},x} = 0, \tag{12}$$

and the total horizontal momentum of the system is,

$$P^\star_x = P^\star_{{\rm A},x} + P^\star_{{\rm B},x} = m \frac{\omega y_0}{\pi} \sin\omega t = m v^\star_{{\rm cm},x} + P^\star_{{\rm hidden},x} = m v^\star_{{\rm cm},x}, \tag{13}$$

recalling eq. (4).

In sum, this example contains no hidden momentum, as defined by eq. (1).

3 Comments

The notion of "hidden" momentum was first introduced by Shockley [3] in an electromechanical example, and essentially all subsequent usage of that term has been in electromechanical examples. There, the subsystems have always been taken to be (A) the electromagnetic fields (B) and everything else. When "hidden" momentum in present in such cases it is an effect at order $1/c^2$, where c is the speed of light in vacuum, in that electromagnetic field momentum is of this order. This has led to the association of "hidden" momentum with "relativistic" effects.

The present example is an all-mechanical version of an electromechanical example in which the electromagnetic field, rather than flowing water, transfers mass/energy from one side of a system to the other [4]. In the electromechanical example hidden momentum is present, according to definition (1), whereas there is no hidden momentum in the present, all-mechanical variant of that example.[3]

Acknowledgment

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[3]Note that if the boundary integral were not present in the definition (1), then both subsystems A and B of the present example would contain "hidden" momentum.

References

[1] D. Vanzella, Private communication (June 29, 2012).

[2] K.T. McDonald, *On the Definition of "Hidden" Momentum* (July 9, 2012),
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[3] W. Shockley and R.P. James, *"Try Simplest Cases" Discovery of "Hidden Momentum" Forces on "Magnetic Currents"*, Phys. Rev. Lett. **18**, 876 (1967),
http://kirkmcd.princeton.edu/examples/EM/shockley_prl_18_876_67.pdf

[4] K.T. McDonald, *"Hidden" Momentum in a Coaxial Cable* (March 28, 2002),
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