1 Problem

In sec. 8 of his 1905 paper introducing special relativity [1], Einstein noted: It is remarkable that the energy and the frequency of a light complex (Lichtkomplex) vary with the state of motion of the observer in accordance with the same law. In that paper he did not interpret the ratio of energy to frequency as the number of photons in the “light complex”, but this notion was likely on his mind. We are led to the appealing conclusion that the number of photons associated with a “light complex” is Lorentz invariant.

Einstein’s argument about a “light complex” was for a finite volume in which the fields were approximated as a monochromatic plane wave (reasonably well realized by a laser beam). Details of the fields near the boundary of the volume were not considered. A variant of Einstein’s argument was given by Zeldovich [2], but again its application to a well-defined volume that contained the fields was not pursued. Avron et al. [3] considered a cavity filled with monochromatic waves in its rest frame, and noted that one can obtain conflicting results as to the number of photons according to an observer moving with respect to the cavity.\(^1\)

Consider a hollow rectangular cavity of perfectly conducting material with inner dimensions \(d_x \times d_y \times d_z\) in its rest frame, where it contains electromagnetic energy \(U\) in a single cavity mode of angular frequency \(\omega\). The number of photons in this classical electromagnetic field is said to be \(N = U/\hbar \omega\), where \(\hbar\) is Planck’s constant \(\hbar/2\pi\). Show that in an inertial frame where the cavity has uniform velocity \(v\), an observer reports the same number of photons. You may assume that the motion is perpendicular to a face of the cavity.

2 Solution

2.1 The Cavity Is at Rest

The standing-wave solutions for electromagnetic fields \(\mathbf{E}\) and \(\mathbf{B}\) of angular frequency \(\omega\) inside a perfectly-conducting, rectangular cavity of extent \(0 < x < d_x\), \(0 < y < d_y\), \(0 < z < d_z\) can be written (in Gaussian units) as,\(^2\)

\(^1\)See also [4].
\[ E_x = E_{0x} \cos k_x x \sin k_y y \sin k_z z e^{-i\omega t}, \]
\[ E_y = E_{0y} \sin k_x x \cos k_y y \sin k_z z e^{-i\omega t}, \]
\[ E_z = E_{0z} \sin k_x x \sin k_y y \cos k_z z e^{-i\omega t}, \]
\[ B_x = B_{0x} \sin k_x x \cos k_y y \cos k_z z e^{-i\omega t}, \]
\[ B_y = B_{0y} \cos k_x x \sin k_y y \cos k_z z e^{-i\omega t}, \]
\[ B_z = B_{0z} \cos k_x x \cos k_y y \sin k_z z e^{-i\omega t}, \]
\]  

where the wave vector \( \mathbf{k} \) is,\(^3\)\(^4\)
\[ \mathbf{k} = (k_x, k_y, k_z) = \left( \frac{l\pi}{d_x}, \frac{m\pi}{d_y}, \frac{n\pi}{d_z} \right), \quad \text{and} \quad k = \frac{\omega}{c}, \]

for any set of integers \( \{l, m, n\} \), and \( c \) is the speed of light in vacuum (and inside the cavity). Further, Faraday’s law,
\[ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \]

implies that \( i\mathbf{B}_0 = \mathbf{k} \times \mathbf{E}_0, \)

so,
\[ \mathbf{B}_0 \cdot \mathbf{E}_0 = 0, \quad |\mathbf{B}_0| = |\mathbf{E}_0|, \quad \text{and} \quad i\mathbf{B}_0 = \frac{(k_y E_{0z} - k_z E_{0y}, k_z E_{0x} - k_x E_{0z}, k_x E_{0y} - k_y E_{0x})}{k}. \]

The free-space Maxwell equations \( \nabla \cdot \mathbf{B} = 0 = \nabla \cdot \mathbf{E} \) imply that,
\[ \mathbf{k} \cdot \mathbf{E}_0 = 0 = \mathbf{k} \cdot \mathbf{B}_0. \]

For each set of integers \( \{l, m, n\} \) there are two orthogonal polarizations of the vectors \( \mathbf{E}_0 \) and \( \mathbf{B}_0 \). As usual, the physical fields are the real parts of eqs. (1)-(2).

The electromagnetic energy stored in the cavity is,
\[ U = \frac{1}{2} \int \frac{|\mathbf{E}|^2 + |\mathbf{B}|^2}{8\pi} d\text{Vol} = \frac{d_xd_yd_z|E_0|^2}{64\pi}, \]

independent of the polarization of the fields, and of the mode indices \( \{l, m, n\} \). The classical number of photons in the cavity (at rest) is therefore,
\[ N = \frac{U}{\hbar\omega} = \frac{d_xd_yd_z|E_0|^2}{64\pi\hbar\omega}. \]

\(^3\)The fields (1)-(2) can be decomposed into eight plane waves with the eight wave vectors \( (\pm l\pi/d_x, \pm m\pi/d_y, \pm n\pi/d_z) \).

\(^4\)Nontrivial cavity modes exist only with two or three of \( \{l, m, n\} \) nonzero, so there is no cavity mode in which the wave vector \( \mathbf{k} \) is parallel to a wall of the cavity. Hence, the arguments of Einstein [1] and of Avron et al. [3] are not strictly applicable to the present example.
2.2 The Cavity Has Velocity $v = v \hat{x}'$.

We now consider the cavity as observed in the $'$ frame, in which it has velocity $v = v \hat{x}'$.

The fields inside the cavity are no longer perceived as standing waves, but rather as the sum of waves traveling in the $+x$ and $-x$ directions: $E = E_+ + E_-, B = B_+ + B_-$, where,

\begin{align*}
E_+ &= (E_{0x} \sin k_y y \sin k_z z, -iE_{0y} \cos k_y y \sin k_z z, -iE_{0z} \sin k_y y \cos k_z z) e^{i(k_z x - \omega t)} / 2, \quad (9) \\
E_- &= (E_{0x} \sin k_y y \sin k_z z, iE_{0y} \cos k_y y \sin k_z z, iE_{0z} \sin k_y y \cos k_z z) e^{-i(k_z x + \omega t)} / 2, \quad (10) \\
B_+ &= (-iB_{0x} \cos k_y y \cos k_z z, B_{0y} \sin k_y y \cos k_z z, B_{0z} \cos k_y y \sin k_z z) e^{i(k_z x - \omega t)} / 2, \quad (11) \\
B_- &= (iB_{0x} \cos k_y y \cos k_z z, B_{0y} \sin k_y y \cos k_z z, B_{0z} \cos k_y y \sin k_z z) e^{-i(k_z x + \omega t)} / 2. \quad (12)
\end{align*}

We also note that the total energies of the left- and right-moving waves are equal, and each equal to $U/2$,

\begin{equation}
U = \int \frac{|E_+ + E_-|^2 + |B_+ + B_-|^2}{8\pi} dVol = \int \frac{|E_+|^2 + |B_+|^2}{8\pi} dVol + \int \frac{|E_-|^2 + |B_-|^2}{8\pi} dVol
= U_+ + U_-, \quad (13)
\end{equation}

since the cross terms in the first integral involve $\int_{0}^{d_x} \cos(2k_x x) dx = 0$.

We now need expressions for the fields (9)-(12) in the $'$ frame, where $x' = \gamma(x - vt)$, $y' = y$, $z' = z$, $t' = \gamma(t + vx/c^2)$, and $\gamma = 1/\sqrt{1-v^2/c^2}$.

The phase of a plane wave is a Lorentz invariant, so we can write,

\begin{align*}
E' &= E'_+ + E'_- = E'_+(y, z) e^{i(k'_{x} + x' - \omega' t')} + E'_-(y, z) e^{i(k'_{x} - x' + \omega' t')}, \quad (14) \\
B' &= B'_+ + B'_- = B'_+(y, z) e^{i(k'_{y} + x' - \omega' t')} + B'_-(y, z) e^{i(k'_{y} - x' + \omega' t')}. \quad (15)
\end{align*}

The Lorentz transformations of the wave 4-vectors ($\omega_{\pm} = k_{x} c, k_{x \pm} c = \pm k_{x} c, k_{y} c, k_{z} c$) are,

\begin{equation}
\omega'_{\pm} = k'_{x} c = \gamma(\omega \pm k_{x} v) = \gamma \omega \left(1 \pm \frac{k_{x} v}{k_{x} c}\right), \quad k'_{x \pm} = \gamma(\pm k_{x} + \omega v/c^2), \quad k'_{y} = k_{y}, \quad k'_{z} = k_{z}. \quad (16)
\end{equation}

Waves that move in the $+x'$ (right) direction have the higher frequency $\omega'_{+}$.

The transformations of the electromagnetic fields are,

\begin{align*}
E_{x}' &= E_{x}, \quad E_{y}' = \gamma \left( E_{y} + \frac{v}{c} B_{z} \right), \quad E_{z}' = \gamma \left( E_{z} - \frac{v}{c} B_{y} \right), \quad (17) \\
B_{x}' &= B_{x}, \quad B_{y}' = \gamma \left( B_{y} - \frac{v}{c} E_{z} \right), \quad B_{z}' = \gamma \left( B_{z} + \frac{v}{c} E_{y} \right), \quad (18)
\end{align*}

To calculate the electromagnetic energy inside the moving cavity at time $t'$ we need,

$$|E'|^2 + |B'|^2$$

$$= |E_x|^2 + |B_x|^2 + \gamma^2 \left[ \left( 1 + \frac{v^2}{c^2} \right) (|E_y|^2 + |E_z|^2 + |B_y|^2 + |B_z|^2) + \frac{4v}{c} Re(E_y B_z^* - E_z B_y^*) \right].$$

The volume of the moving cavity is $\frac{d_x d_y d_z}{\gamma}$, due to the Lorentz contraction. Also, the cross terms due to fields with different frequencies will vanish in the calculation the time-average energy stored in the moving cavity, so we can write,

$$\langle U' \rangle = \langle U'_+ \rangle + \langle U'_- \rangle,$$

where,

$$\langle U'_\pm \rangle = \frac{1}{2} \int \frac{|E_\pm|^2 + |B_\pm|^2}{8\pi} dVol' = \frac{d_x d_y d_z}{256\pi \gamma} \left\{ |E_{0x}|^2 + |B_{0x}|^2 + \gamma^2 \left[ \left( 1 + \frac{v^2}{c^2} \right) (|E_{0y}|^2 + |E_{0z}|^2 + |B_{0y}|^2 + |B_{0z}|^2) + \frac{4v}{c} Re(E_{0y} B_{0z}^* - E_{0z} B_{0y}^*) \right] \right\}$$

$$= \frac{\gamma d_x d_y d_z}{256\pi} \left\{ \left( 1 - \frac{v^2}{c^2} \right) (|E_{0x}|^2 + |B_{0x}|^2) + \left( 1 + \frac{v^2}{c^2} \right) (|E_{0y}|^2 + |E_{0z}|^2 + |B_{0y}|^2 + |B_{0z}|^2) \right\}$$

$$\pm \frac{4v}{c} Re(i E_{0y} B_{0z}^* - i E_{0z} B_{0y}^*)$$

Using eq. (5) and noting that $\mathbf{k} \cdot \mathbf{E}_0^* = 0$, we have,

$$|E_{0x}|^2 + |B_{0x}|^2 - |E_{0y}|^2 - |E_{0z}|^2 - |B_{0y}|^2 - |B_{0z}|^2$$

$$= 2 |E_0|^2 - 2 |E_{0y}|^2 - 2 |E_{0z}|^2 - 2 |B_{0y}|^2 - 2 |B_{0z}|^2$$

$$= 2 |E_{0x}|^2 + \frac{2}{k_x^2} \{ k_x^2 |E_{0x}|^2 + k_x^2 |E_{0z}|^2 - 2 k_x k_z Re(E_{0x} E_{0z}^*) \}$$

$$- \frac{2}{k_y^2} \{ k_y^2 |E_{0y}|^2 + k_y^2 |E_{0z}|^2 - 2 k_y k_z Re(E_{0y} E_{0z}^*) \}$$

$$= 2 |E_{0x}|^2 - \frac{2}{k_x^2} \{ k_x^2 |E_{0x}|^2 + k_x^2 |E_0|^2 - 2 k_x Re(E_{0x} (k_x E_{0x}^* + k_y E_{0y}^* + k_z E_{0z}^*)) \}$$

$$= - \frac{2k_x^2}{k_x^2} |E_0|^2.$$

and similarly,

$$i E_{0y} B_{0y}^* - i E_{0y} B_{0z}^* = - \frac{k_x}{k} |E_0|^2.$$

Thus, recalling eq. (7),

$$\langle U'_\pm \rangle = \frac{\gamma d_x d_y d_z}{128\pi} \left( 1 \pm \frac{k_x v}{k c} \right)^2 |E_0|^2 = \gamma \left( 1 \pm \frac{k_x v}{k c} \right)^2 \frac{U}{2}.$$

4
The total field energy $U'$ inside the moving cavity is independent of time,

$$\langle U' \rangle = U' = \gamma \left( 1 + \frac{k_x v^2}{k^2 c^2} \right) U.$$ (25)

The numbers of classical photons in the moving cavity are, recalling eqs. (8) and (16),

$$N'_\pm = \frac{\langle U'_\pm \rangle}{\hbar \omega_\pm} = \gamma \left( 1 \pm \frac{k_x v}{k c} \right)^2 \frac{U}{2} \left( 1 \pm \frac{k_x v}{k c} \right).$$ (26)

While there are more photons of higher frequency ($\omega_+$, right-moving) than of lower frequency ($\omega_-$, left-moving) in the moving cavity, the total number $N' = N'_+ + N'_- = N$ of photons observed at a time $t'$ is the same as in a frame where the cavity is at rest (at a time $t$).\textsuperscript{5,6}

### 2.3 Is the Field Energy Part of the Rest Energy of the System?

A side issue of possible interest is the question of whether the field energy $U$ can/should be considered as contributing to the “rest” energy of the system, such that,

$$U_0 = Mc^2 + U,$$ (27)

in the rest frame of the cavity, whose rest mass without fields inside is $M$.

If so, then in the frame where the cavity has velocity $v = v \hat{x}$, the energy of the system should be,

$$U' = \gamma U_0 = \gamma (Mc^2 + U).$$ (28)

However, in this frame the cavity itself has energy,

$$U'_{\text{cavity}} = \gamma Mc^2,$$ (29)

and the fields have energy,

$$U' = \gamma U \left( 1 + \frac{k_x^2 v^2}{k^2 c^2} \right),$$ (30)

so that the actual energy of the system is,

$$U' = \gamma \left[ Mc^2 + U \left( 1 + \frac{k_x^2 v^2}{k^2 c^2} \right) \right],$$ (31)

rather than expression (28).

Missing from the above analysis is the energy associated with the stress on the walls of the cavity due to the “radiation pressure” of the fields inside. Presumably, if these stresses, and their Lorentz transformations, were included in the analysis, a meaningful rest energy could be defined for the system. For a simpler example of this, see [5].

\textsuperscript{5}For observations in the $'$ frame at time $t' = 0$, the corresponding times of observation in the rest frame are $t = 0$ for the left wall at $x = 0$ and $t = -vd_x/c^2$ for the right wall at $x = d_x$. The $'$ observer counts some photons near the right wall as being right-moving at time $t' = 0$ ($t < 0$), while the rest frame observer will count these as left-moving at the later time $t = 0$ after they have reflected off the right wall. That is, the $'$ observer counts more right-moving (higher-energy) than left-moving photons.

\textsuperscript{6}Thanks to William Celmaster for spotting several typos in a previous version of this section.
A Appendix: Are the Cavity Photons Localized?

The cavity photons are “localized” in the sense of being “trapped” inside the cavity. A cavity photon of energy \( E = \hbar \omega \) has no momentum, \( \mathbf{P} = 0 \), and so can be said to have effective mass
\[
m = \sqrt{E^2 - c^2 P^2 / c^2} = \frac{\hbar \omega}{c^2}.
\]
The cavity photons are “virtual photons,” with nonzero mass.

However, people also speak of “Anderson localization” [6] of quantum waves/particles in disordered media when ordinary diffusion is absent due to the localizing effect of multiple scattering. Anderson localization of light was perhaps first discussed in [7], and is considered to have been observed (see, for example, [8], but apparently some reports of this must be regarded with skepticism (see, for example, [9]).

References

http://kirkmcd.princeton.edu/examples/EM/einstein_ap_17_891_05.pdf
http://kirkmcd.princeton.edu/examples/EM/einstein_ap_17_891_05_english.pdf


http://kirkmcd.princeton.edu/examples/CM/john_prl_53_2169_84.pdf
