What Is the Vector Potential of a Uniform Magnetic Field?

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1 Introduction

We consider a magnetic field $\mathbf{B} = B\,\hat{\mathbf{z}}$ that is (at least approximately) uniform in some region of space, but possibly time dependent.

In general, the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ can be related to potentials V and $\mathbf{A}$ by

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t}\,, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{1}$$

in SI units. We assume that the sources of the uniform magnetic field $\mathbf{B}$ do not involve a nonzero electric charge density, and we might naïvely assume that the scalar potential is $V = 0$.

Vector potentials $\mathbf{A}$ for the uniform magnetic field $\mathbf{B}$ can be expressed in (x, y, z) coordinates, in the region of uniform $\mathbf{B}$, as

$$\mathbf{A} = -\frac{ayB}{b}\,\hat{\mathbf{x}} + \frac{(b-a)xB}{b}\,\hat{\mathbf{y}}, \tag{2}$$

for any real numbers a, b with nonzero b,[1,2] since for the vector potential of eq. (2),

$$\boldsymbol{\nabla} \times \mathbf{A} = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}} = \left(\frac{(b-a)B}{b} + \frac{aB}{b}\right)\hat{\mathbf{z}} = B\,\hat{\mathbf{z}} = \mathbf{B}. \tag{3}$$

If we are only concerned with the magnetic field $\mathbf{B}$, all of the infinite set of vector potentials (2) are equally valid. But, if we also consider the electric field $\mathbf{E}$, induced by a time-dependent D according to Faraday's law,

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{4}$$

the infinite set of vector potentials described by eq. (2) would imply an infinite set of different electric fields if the scalar potential V is indeed zero for all the variants of eq. (2). However, a physical example can have only one electric field, so we infer that the scalar potential in nonzero in general, such that eq (1) leads to the same electric field for all versions of eq. (2).

We now illustrate how symmetries of two example configurations can select the vector potentials corresponding to zero scalar potential.

[1]Note that eq. (2) satisfies the Coulomb-gauge condition that $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, as well as the Lorenz-gauge condition that $\boldsymbol{\nabla} \cdot \mathbf{A} = -(1/c^2)\,\partial V/\partial t$ for $V = 0$.

[2]We omit possible constant terms in eq. (2).

2 Infinite Solenoid

We first consider the case of an infinite solenoid of radius R, whose symmetry axis is the z-axis. The magnetic field is uniform, $B\,\hat{\mathbf{z}}$, inside the solenoid and "zero" outside it

According to Faraday's law, eq. (4), the possible time dependence of $\mathbf{B} = B\,\hat{\mathbf{z}}$ can only induce x- and y- components of the electric field $\mathbf{E}$, and further, $\mathbf{E}$ must be azimuthally symmetric for an infinite solenoid; $\mathbf{E} = E_\phi\,\hat{\boldsymbol{\phi}}$ in cylindrical coordinates (r, ϕ, z).

We can deduce $\mathbf{E}$ using the integral form of Faraday's law,

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_{\mathbf{B}}}{dt}\,, \tag{5}$$

where the magnetic flux $\Phi_{\mathbf{B}}$ is, for a loop of radius r about the z-axis,

$$\Phi_{\mathbf{B}} = \int \mathbf{B} \cdot d\mathbf{Area} = B \begin{cases} \pi r^2 & (r < R), \\ \pi R^2 & (r > R), \end{cases} \tag{6}$$

From eqs. (5) and (6) we have, for the azimuthal electric field E_ϕ,

$$\oint \mathbf{E} \cdot d\mathbf{l} = 2\pi r E_\phi = -\frac{d\Phi_{\mathbf{B}}}{dt} = -\dot{B} \begin{cases} \pi r^2 & (r < R), \\ \pi R^2 & (r > R). \end{cases} \tag{7}$$

If we now assume that the scalar potential V is zero, we have

$$E_\phi = -\frac{\partial A_\phi}{\partial t} = -\dot{B} \begin{cases} r/2 & (r < R), \\ R^2/2r & (r > R), \end{cases} \tag{8}$$

where $\dot{B} = dB/dt$. Hence, the potentials have the form (omitting possible constant terms)

$$\mathbf{A} = \frac{B\,\hat{\boldsymbol{\phi}}}{2} \begin{cases} r & (r < R), \\ R^2/r & (r > R), \end{cases} \qquad V = 0, \tag{9}$$

where $\hat{\boldsymbol{\phi}} = -y\,\hat{\mathbf{x}}/r + x\,\hat{\mathbf{y}}/r$.

The induced electric field $\mathbf{E}$ of eq. (8) is nonzero for $r > R$ where the magnetic field $\mathbf{B}$ is nominally zero (although $\mathbf{B}$ is nonzero everywhere outside any finite solenoid). This is an example of "nonlocal" behavior in classical electrodynamics.

We can obtain other potentials for this example via the *restricted* gauge transformation

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi, \qquad V' = V - \frac{\partial \chi}{\partial t}\,, \tag{10}$$

using any function χ that obeys $\nabla^2\chi = 0$, such that the potentials $\mathbf{A}'$ and V' imply the same $\mathbf{E}$ and $\mathbf{B}$ fields according to eq. (1) as do $\mathbf{A}$ and V (and the new potentials are also in the Coulomb gauge).

A famous example, due to Landau, is to consider $\chi = Bxy/2$ (or $-Bxy/2$), for which

$$\boldsymbol{\nabla}\chi = \frac{By}{2}\,\hat{\mathbf{x}} + \frac{Bx}{2}\,\hat{\mathbf{y}}, \qquad \frac{\partial\chi}{\partial t} = \frac{\dot{B}xy}{2}\,. \tag{11}$$

The transformed potentials are

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi = B\begin{cases} x\,\hat{\mathbf{y}} & (r < R), \\ R^2\,\hat{\boldsymbol{\phi}}/2r + y\,\hat{\mathbf{x}}/2 + x\,\hat{\mathbf{y}}/2 & (r > R), \end{cases} \qquad V' = -\frac{\partial\chi}{\partial t} = -\frac{\dot{B}xy}{2}\,. \tag{12}$$

3 A Pair of Current Sheets

Another idealized, but calculable, configuration involving a uniform magnetic field $\mathbf{B} = B\,\hat{\mathbf{z}}$ is a pair of current sheets in, say, x-z planes at $y = \pm d$, with uniform current density $\mathbf{J}$ in the $-x$ direction for $y = d$, and in the $+x$ direction for $y = -d$. Here, the magnetic field is zero for $|y| > d$.

If B is time dependent (but always uniform in space), an electric field $\mathbf{E}$ is induced, and again from Faraday's law it follows that $\mathbf{E}$ has only x- and y-components. Use of the integral form of Faraday's law for various rectangular loops only determines E_x, but not E_y, so we infer that $E_y = 0$.

Consider a rectangular loop in the x-y plane with one edge at $y = 0$ and the opposite edge at some nonzero value of y. The length of the loop in x is l, so for $|y| < d$ Faraday's law tells us that

$$\oint \mathbf{E}\cdot d\mathbf{l} = l[E_x(y=0) - E_x(y)] = -\frac{d\Phi_{\mathbf{B}}}{dt} = -ly\dot{B}, \tag{13}$$

$$E_x(y) = E_x(y=0) + y\dot{B}, \tag{14}$$

The (anti)symmetry of this configuration indicates that $E_x(y=0) = 0$.

If we now consider a rectangular loop whose outer edge is at $y > |d|$, the magnetic flux is only $d\,l\,B$, and we learn that

$$E_x(y) = \dot{B}\begin{cases} y & (|y| < d), \\ d\ \mathrm{sgn}(y) & (|y| > d), \end{cases} \tag{15}$$

where $\mathrm{sgn}(y) = 1$ for $y > 0$ while $\mathrm{sgn}(y) = -1$ for $y < 0$. Again, the induced electric field is nonzero in the region where the magnetic field is zero.

The potentials corresponding to this electric field, again assuming $V = 0$ so that $\mathbf{E} = -\partial\mathbf{A}/\partial t$, are

$$\mathbf{A} = -B\,\hat{\mathbf{x}}\begin{cases} y & (|y| < d), \\ d\,\mathrm{sgn}(y) & (|y| > d), \end{cases} \qquad V = 0, \tag{16}$$

plus possible constant terms.

4 Comments

The vector potentials of eqs. (9) and (16) are those calculated in the quasistatic approximation,

$$V(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\, d^3\mathbf{x}', \tag{17}$$

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\, d^3\mathbf{x}', \tag{18}$$

where ρ and $\mathbf{J}$ are the charge and current densities, to the retarded potentials in the Lorenz gauge,

$$V^{(\mathrm{L})}(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}\, d^3\mathbf{x}' \qquad \text{where} \qquad t' = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \tag{19}$$

$$\mathbf{A}^{(\mathrm{L})}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}\, d^3\mathbf{x}'. \tag{20}$$

The scalar potential V of eq. (17) is the "standard" Coulomb-gauge scalar potential,

$$V^{(\mathrm{C})}(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\, d^3\mathbf{x}', \tag{21}$$

while the vector potential of eq. (18) is the static limit of the "standard" Coulomb-gauge vector potential, which in general has the (intricate) form

$$\mathbf{A}^{(\mathrm{C})}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}_{\mathrm{rot}}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\, d^3\mathbf{x}', \tag{22}$$

where $\mathbf{J} = \mathbf{J}_{\mathrm{rot}} + \mathbf{J}_{\mathrm{irr}}$ with $\boldsymbol{\nabla}\cdot\mathbf{J}_{\mathrm{rot}} = 0$ and $\boldsymbol{\nabla}\times\mathbf{J}_{\mathrm{irr}} = 0$ and, in general,

$$\mathbf{J}_{\mathrm{rot}}(\mathbf{x},t) \;=\; \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\int \frac{\mathbf{J}(\mathbf{r}',t)}{4\pi\,|\mathbf{x}-\mathbf{x}'|}\, d^3\mathbf{x}'. \tag{23}$$

In the static limit, $\boldsymbol{\nabla}\cdot\mathbf{J} = 0$, in which case $\mathbf{J}_{\mathrm{rot}} = \mathbf{J}$, and the "standard" Coulomb-gauge vector potential (22) has the form of eq. (18).