# Force between Two Uniformly Magnetized Spheres

Kirk T. McDonald
*Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544*
(March 23, 2016; updated January 25, 2017)

## 1 Problem

Show that the force, and also the torque, between two uniformly magnetized spheres is the same as that between two "point" magnetic dipoles of the same total magnetic moments, located at the centers of the spheres.

*This problem was suggested by Boyd Edwards [1, 2]. It appears as prob. 6.6, p. 284 of [3].*

## 2 Solution

The solution will rely on the mean-value theorem for (harmonic) vector fields that obey Laplace's equation inside a spherical volume.[1] Namely, the average of such a field over the sphere is equal to the value of the field at its center.[2] This theorem applies to static electric and magnetic fields inside source-free spheres.

Also, we note that the field exterior to a uniformly polarized (electric or magnetic) sphere is the same as that due to a point dipole at the center of the sphere with that same total dipole moment as the uniform sphere.[3] A consequence is that the force and torque on two uniformly magnetized spheres is the same as that on a point magnetic dipole plus a uniformly magnetized sphere, with the same total magnetic moments.

Section 2.3 gives an alternative solution via the Maxwell stress tensor. An argument based on Newton's 3rd law of action and reaction, together with the fact that the field external to a uniformly magnetized sphere is that of a point dipole, has been given in [1, 2].

### 2.1 Force

A direct computation of the force on a uniformly magnetized sphere is cumbersome. Appendix B.2 demonstrates such a computation for one special case, when the moments of the magnetized sphere and point dipole are both parallel to their line of centers, with the result that this force is equal to that between two point dipoles of the same strength.

For a more general argument, we use an energy method (as suggested by B. Edwards). The force on a rigid, uniform sphere can be deduced from the gradient (with respect to the position of the center of the sphere) of its interaction energy $U$,

$$\mathbf{F} = -\boldsymbol{\nabla} U. \tag{1}$$

[1] This theorem was first deduced by W. Thomson, p. 225 of [4].
[2] See, for example, eq. (4.19) of [5], prob. 5.59 of [6], and [7].
[3] See Appendix A and, for example, sec. 5.10 of [5] and exs. 4.2, 6.1 of [6].

We recall that the interaction energy of a magnetic dipole (of fixed magnitude) with an external magnetic field $\mathbf{B}$ has the form $U = -\mathbf{m} \cdot \mathbf{B}$.[4] Then, the interaction energy of a uniformly magnetized sphere with total moment $\mathbf{M}$ due to a external magnetic dipole $\mathbf{m}$ is,

$$U = -\int d\mathbf{M} \cdot \mathbf{B_m} = -\mathbf{M} \cdot \int \mathbf{B_m}\, d\mathrm{Vol} = -\mathbf{M} \cdot \mathbf{B_m}(\text{center of } \mathbf{M}), \tag{2}$$

which is the same as the interaction energy between the point dipole $\mathbf{m}$ and a point dipole $\mathbf{M}$ at the location of the center of the uniformly magnetized sphere.

It immediately follows that the force on the uniformly magnetized sphere is that same as if it were a point magnetic dipole located at the center of that sphere.

## 2.2 Torque

We recall that the torque on a point magnetic dipole $\mathbf{m}$ due to an external magnetic field $\mathbf{B}$ is,

$$\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}. \tag{3}$$

See, for example, sec. 5.7 of [5] and sec. 6.1.2 of [6].

Then, the total torque on a uniformly magnetized sphere of magnetic moment $\mathbf{M}$ is,

$$\boldsymbol{\tau} = \int d\mathbf{M} \times \mathbf{B_m} = \mathbf{M} \times \int \mathbf{B_m}, d\mathrm{Vol} = \mathbf{M} \times \mathbf{B_m}(\text{center of } \mathbf{M}), \tag{4}$$

which is the same as the torque on a point dipole $\mathbf{M}$ located at the center of the uniformly magnetized sphere.

## 2.3 Argument via the Maxwell Stress Tensor

*This section follows comments by David Griffiths and by J. Castro Paredes.*

The electromagnetic force on an object can be computed via the Maxwell stress tensor (arts. 630-645 of [8]) on any surface exterior to the object that has no matter within it except the object in question. The stress tensor depends on the total fields on that surface, which in the present case are the sum of the magnetic fields exterior to two uniformly magnetized spheres (or one such sphere and a point magnetic dipole)

Since the magnetic field exterior to both of a pair of uniformly magnetized spheres is the same as the field due to "equivalent" point dipoles at the locations of the centers of the spheres with the same magnetic moments as the spheres, we immediately infer that the force on either of these spheres is the same as on a pair of equivalent magnetic moments.

Also, since the torque about the center of either of the spheres can be computed by integrating the cross product of a radius vector with the force elements inferred from the Maxwell stress tensor, the torque is also the same as that associated with the equivalent magnetic dipoles.

[4]See, for example, sec. 5.7 of [6].

# A Appendix A: Fields of a Uniformly Polarized Sphere

The fields of a uniformly polarized sphere may have been first discussed by W. Thomson (Lord Kelvin) in the footnote on p. 476 of [9].[5] He considered two spheres with uniformly distributed mass $\pm m$ (supposing that negative mass exists), whose centers are displaced by small distance $\mathbf{d}$. As follows from an argument of Newton for $1/r^2$ force laws (p. 218 of [11]), the force/field outside each sphere is the same as if the mass were concentrated at its center. Hence, the exterior field is the same as that of an gravitational dipole $\mathbf{p} = m\mathbf{d}$ (supposing that such an entity exists).[6]

The (radial) force/field at an interior point at distance $r$ from the center of a sphere of radius $R$ is $mr/R^3$, so the force/field at this point in case of two slightly displaced spheres of opposite mass is, by similar triangles, antiparallel to the dipole moment $\mathbf{p}$ of the system, with magnitude $(mr/R^3)(d/r) = p/R^3$ (for unit gravitational constant), independent of $r$. That is, the interior force/field is uniform.

The interior can be thought of as containing a uniform density $\mathbf{P}$ of dipoles, such that $\mathbf{p} = 4\pi\mathbf{P}R^3/3$, and hence the interior force/field has the value $-3\mathbf{P}/4\pi$, in terms of the polarization density $\mathbf{P}$.

Thomson confirmed these results on p. 492 of [9] by an expansion in spherical harmonics of the potential of the force/field (now considered to be magnetic rather than gravitational). He refers to his paper [12] of 1843 for the introduction of this technique in the context of heat flow.

# B Appendix B: Direct Force Calculations

This problem recalls the case of Newtonian gravity, where famously the force between two uniform spherical masses (or shells) is the same as if all the mass were concentrated at their centers. Hence, we first review the gravitational lore.

## B.1 Gravitational Attraction between a Point and a Sphere

Suppose the attractive force of gravity between two point masses $m$ and $M$ is along their line of centers, and has the form,

$$F = \frac{G_n m M}{r^n}, \tag{5}$$

where $G_n$ is a constant that depends on the exponent $n$.

To determine whether the force of gravity between two spherical masses (of uniform mass density) has the same form as eq. (5), it suffices to consider that case of a point mass $m$ and a spherical mass $M$ of radius $a$ and mass density $\rho$ where $M = 4\pi\rho a^3/3$. We take mass $M$

[5] That footnote, probably written in 1872 (the year of the first edition of [9], since it is mentioned in sec. XXXII, p. 492, dated May 29, 1872), is for the reprint of Thomson's 1851 paper [10]. Page numbers are for the 1884 edition of [9].

[6] Thomson did not explicitly state that the exterior field is that of a dipole in the footnote on p. 476 of [9], but only in the related discussion on p. 497, where he was considering magnetic polarization.

to be centered on the origin in a spherical coordinate system $(r, \theta, \phi)$, with point mass $m$ at $r = R$ on the $z$-axis. For what values of $n$ does the force between these two masses (which is directed along the $z$-axis have the form $G_n mM/R^n$?

This force can be computed as,

$$\begin{aligned} F_z &= 2\pi G_n m\rho \int_0^a r^2\,dr \int_{-1}^1 d\cos\theta\, \frac{R - r\cos\theta}{(R^2 - 2rR\cos\theta + r^2)^{(n+1)/2}} \\ &= 2\pi G_n m\rho R^{3-n} \int_0^{a/R} x^2\,dx \int_{-1}^1 dy\, \frac{1 - xy}{(1 - 2xy + x^2)^{(n+1)/2}}\,, \end{aligned} \tag{6}$$

with the substitutions $x = r/R$ and $y = \cos\theta$.

### B.1.1 $n = 2$, Newtonian Gravity

This is the form of Newtonian gravity, which was determined by Newton on p. 218 of [11] via a geometrical argument to have the property that the force of attraction of a sphere is the same as if all the mass were concentrated at it center. That is, $G_2 = G$, Newton's gravitational constant.

For $n = 2$, we use Dwight 191.03 and 191.13 [13] in eq. (6) to find,

$$\begin{aligned} F_z &= 2\pi G_2 m\rho R \int_0^{a/R} x^2\,dx \left[\frac{1}{x(1 - 2xy + x^2)^{1/2}} - \frac{1}{2x}\left((1 - 2xy + x^2)^{1/2} + \frac{1 + x^2}{(1 - 2xy + x^2)^{1/2}}\right)\right]_{-1}^{1} \\ &= 2\pi G_2 m\rho R \int_0^{a/R} x\,dx \left[\left(\frac{1}{1 - x} - \frac{1}{1 + x}\right)\left(1 - \frac{1 + x^2}{2}\right) + x\right] \\ &= 4\pi G_2 m\rho R \int_0^{a/R} x^2\,dx = \frac{G_2 m}{R^2}\frac{4\pi\rho a^3}{3} = \frac{G_2 mM}{R^2} = \frac{GmM}{R^2}\,. \end{aligned} \tag{7}$$

### B.1.2 $n = -1$, Springlike Gravity

For $n = -1$ (corresponding to an ideal spring or zero unstretched length), we use Dwight 90.2, 91.2 and 620.1 in eq. (6) to find,

$$F_z = 4\pi G_{-1} m\rho R^4 \int_0^{a/R} x^2\,dx = G_{-1} mR\frac{4\pi\rho a^3}{3} = G_{-1} mMR. \tag{8}$$

Apparently [14], $n = 2$ and $-1$ are the only cases for which a central force on a uniform sphere (due to an exterior point source) is the same as if all the mass were at its center.

### B.1.3 $n = 3$

For an example where the "gravitational" force between point mass and a uniform sphere is not the same as between the point mass and a second point mass at the center of the sphere, we consider $n = 3$.

We use Dwight 90.2, 91.2 and 620.1 in eq. (6) to find,

$$
\begin{aligned}
F_z &= 2\pi G_3 m\rho \int_0^{a/R} x^2\, dx \left[\frac{1}{2x(1-2xy+x^2)} - \frac{1}{4x}\left(\ln(1-2xy+x^2) + \frac{1+x^2}{(1-2xy+x^2)}\right)\right]_{-1}^{1} \\
&= \pi G_3 m\rho \int_0^{a/R} x\, dx \left[\left(\frac{1}{(1-x)^2} - \frac{1}{(1+x)^2}\right)\left(1 - \frac{1+x^2}{2}\right) + \ln\frac{1+x}{1-x}\right] \\
&= \pi G_3 m\rho \int_0^{a/R} \left(2x^2 + x\ln\frac{1+x}{1-x}\right) dx = \pi G_3 m\rho \frac{2a^3}{3R^2} + \pi G_3 m\rho \left[x + \frac{x^2-1}{2}\ln\frac{1+x}{1-x}\right]_0^{a/R} \\
&= \pi G_3 m\rho \left[\frac{2a^3}{3R^2} + \frac{a}{R} + \frac{1}{2}\left(\frac{a^2}{R^2} - 1\right)\ln\frac{1+a/R}{1-a/R}\right] \\
&\approx \pi G_3 m\rho \left(\frac{4a^3}{3R^3} + \frac{2a^5}{15R^5}\right) = \frac{G_3 mM}{R^3}\left(1 + \frac{a^2}{10R^2}\right) \neq \frac{G_3 mM}{R^3}\,. \qquad (9)
\end{aligned}
$$

This shows that it is not obvious that the non-central force between two uniformly magnetized spheres (ignoring gravity) will be the same as for "point" magnetic dipoles of the same total magnetic moment.

## B.2 Two Uniformly Magnetized Spheres

The magnetic field exterior to a uniformly magnetized sphere is the same as for a point magnetic dipole at the center of the sphere with the same total magnetic moment $\mathbf{m}$,

$$\mathbf{B}_{\rm ext} = 3\frac{(\mathbf{m}\cdot\mathbf{r})\,\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}\,, \qquad (10)$$

in Gaussian units and with the origin at the center of the sphere. Hence, when considering the force of the force between two uniformly magnetized spheres, if suffices to consider the force between a point magnetic dipole and a uniformly magnetized sphere.

The force on a "point" magnetic dipole $\mathbf{m}$ in an external magnetic field $\mathbf{B}$ can be deduced from the interaction energy $U = -\mathbf{m}\cdot\mathbf{B}$ as,

$$\mathbf{F} = -\boldsymbol{\nabla} U = \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B}), \qquad (11)$$

where the gradient operator $\boldsymbol{\nabla}$ acts only on the magnetic field $\mathbf{B}$. The force acting a point magnetic dipole $\mathbf{M}$ due to another point magnetic dipole $\mathbf{m}$ is,

$$
\begin{aligned}
\mathbf{F} &= \boldsymbol{\nabla}\left(3\frac{(\mathbf{m}\cdot\mathsf{r})(\mathbf{M}\cdot\mathsf{r})}{\mathsf{r}^5} - \frac{\mathbf{m}\cdot\mathbf{M}}{\mathsf{r}^3}\right) \\
&= 3\frac{(\mathbf{m}\cdot\mathsf{r})\mathbf{M} + (\mathbf{M}\cdot\mathsf{r})\mathbf{m} + (\mathbf{m}\cdot\mathbf{M})\,\mathsf{r}}{\mathsf{r}^5} - 15\frac{(\mathbf{m}\cdot\mathsf{r})(\mathbf{M}\cdot\mathsf{r})\,\mathsf{r}}{\mathsf{r}^7}\,, \qquad (12)
\end{aligned}
$$

where $\mathsf{r}$ points from $\mathbf{m}$ to $\mathbf{M}$.

As an example, if both $\mathbf{m}$ and $\mathbf{M}$ point in the same direction, and both parallel (antiparallel) to $\mathsf{r}$, then the force $F$ on $\mathbf{M}$ is attractive (repulsive), with magnitude,

$$F = \frac{6mM}{R^4}\,, \qquad (13)$$

where $R$ is the distance between the dipoles. A particular example is a uniformaly magnetized sphere with moment $m\,\hat{\mathbf{z}}$, centered at height $h$ above a superconducting plane at $z = 0$. Then, the image has moment $-m\,\hat{\mathbf{z}}$, so the levitating force on the sphere is $3m^2/2h^4$.

We now consider a uniformly magnetized sphere of radius $a$ and total magnetic moment $\mathbf{M} = M\,\hat{\mathbf{z}}$ whose center is at distance $R$ from a point magnetic dipole with magnetic moment $\mathbf{m} = \pm m\,\hat{\mathbf{z}}$. As in sec. A.1, we take the origin at the center of the sphere and suppose the point dipole to be on the positive $z$-axis; $\mathbf{R} = -R\,\hat{\mathbf{z}}$. Is this system equivalent to two point magnetic dipoles, $\mathbf{M} = M\,\hat{\mathbf{z}}$ at the origin and $\mathbf{m} = \pm m\,\hat{\mathbf{z}}$ at $R\,\hat{\mathbf{z}}$? If so, the force on the magnetized sphere at the origin should be in the positive $z$-direction, with magnitude given by eq. (13).

To check this, we will integrate the force on the sphere in a spherical coordinate system $(r, \theta, \phi)$. As before, we consider small volume elements inside the magnetized sphere, which are at distance $\mathsf{r} = \sqrt{R^2 - 2rR\cos\theta + r^2}$ from the point dipole, such that,

$$\mathsf{r}_z = -(R - r\cos\theta), \qquad \mathbf{m}\cdot\mathsf{r} = \mp m(R - r\cos\theta), \qquad \mathbf{M}\cdot\mathsf{r} = -M(R - r\cos\theta). \tag{14}$$

Writing the density of uniform magnetization in the sphere (of radius $a$) as $\rho = 3a^3M/4\pi$, the force of point magnetic dipole $\mathbf{m}$ on the sphere has only the $z$-component,

$$\begin{aligned} F_z &= 2\pi m\rho \int_0^a r^2\,dr \int_{-1}^1 d\cos\theta \left( \frac{-9(R - r\cos\theta)}{(R^2 - 2rR\cos\theta + r^2)^{5/2}} + \frac{15(R - r\cos\theta)^3}{(R^2 - 2rR\cos\theta + r^2)^{7/2}} \right) \\ &= \frac{6\pi m\rho}{R} \int_0^{a/R} x^2\,dx \int_{-1}^1 dy \left[ \frac{-3(1 - xy)}{(1 - 2xy + x^2)^{5/2}} + 5\frac{(1 - xy)^3}{(1 - 2xy + x^2)^{7/2}} \right]. \end{aligned} \tag{15}$$

For the $y$-integral, we follow sec. 190 of Dwight in defining (for fixed $x$),

$$a = 1 + x^2, \quad b = -2x, \quad c = 1 - x^2, \qquad Y = a + by = 1 - 2xy + x^2, \quad 1 - xy = \frac{Y + c}{2}. \tag{16}$$

Then, the $y$-integral can be written as,

$$\begin{aligned} I_y &= \int_{-1}^1 dy \left[ \frac{-3(Y + c)}{2Y^{5/2}} + 5\frac{Y^3 + 3Y^2c + 3Yc^2 + c^3}{8Y^{7/2}} \right] \\ &= \frac{1}{8}\int_{-1}^1 dy \left[ \frac{5}{Y^{1/2}} + \frac{-12 + 15c}{Y^{3/2}} + \frac{c(-12 + 15c)}{Y^{5/2}} + \frac{5c^3}{Y^{7/2}} \right] \\ &= \frac{1}{8x}\left[ -5Y^{1/2} + \frac{-12 + 15c}{Y^{1/2}} + \frac{c(-12 + 15c)}{3Y^{3/2}} + \frac{5c^3}{5Y^{5/2}} \right]_{-1}^{1} \\ &= \frac{1}{4}\left[ 5 + \frac{-12 + 15c}{c} + \frac{(4 - c)(-4 + 5c)}{c^2} + \frac{16 - 12c + c^2}{c^2} \right] \\ &= 4, \end{aligned} \tag{17}$$

using Dwight 191. Then, the force between the point dipole $m$ and the magnetized sphere is, using Dwight 142.1 and 142.2,

$$F_z = \frac{24\pi m\rho}{R}\int_0^{a/R} x^2\,dx = \frac{24\pi m\rho a^3}{3R^4} = \frac{6mM}{R^4}\,, \tag{18}$$

in agreement with the result (13) for point magnetic dipoles $\mathbf{m}$ and $\mathbf{M}$.

The case of uniformly magnetized spheres with moments parallel to one another but perpendicular to their line of centers, has been treated in [15] via the so-called shape function.

## References


[1] B. Edwards *et al.*, *Dynamical interactions between two uniformly magnetized spheres*, Eur. J. Phys. **38**, 015205 (2017), http://kirkmcd.princeton.edu/examples/EM/edwards_ejp_38_015205_17.pdf

[2] B. Edwards *et al.*, *Interactions between two uniformly magnetized spheres*, Am. J. Phys. **85**, 130 (2017), http://kirkmcd.princeton.edu/examples/EM/edwards_ajp_85_130_17.pdf

[3] P.C. Clemmow, *An Introduction to Electromagnetic Theory* (Cambridge U. Press, 1973), http://kirkmcd.princeton.edu/examples/EM/clemmow_p284.pdf

[4] W. Thomson, *Demonstration of a Fundamental Proposition in the Mechanical Theory of Electricity*, Camb. Dublin Math. J. **4**, 223 (1845), http://kirkmcd.princeton.edu/examples/EM/thomson_cmj_4_223_45.pdf

[5] J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, 1999), http://kirkmcd.princeton.edu/examples/EM/jackson_ce3_99.pdf

[6] D.J. Griffiths, *Introduction to Electrodynamics*, 4th ed. (Pearson, 2004), http://kirkmcd.princeton.edu/examples/EM/griffiths_em3.pdf

[7] B. Y.-K. Hu, *Averages of static electric and magnetic fields over a spherical region: a derivation based on the mean-value theorem*, Am. J. Phys. **68**, 1058 (2000), http://arxiv.org/abs/physics/0002021

[8] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 2 (Clarendon Press, Oxford, 1873), http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v2_73.pdf
Vol. 2, 3rd ed. (Clarendon Press, 1892), http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v2_92.pdf

[9] W. Thomson, *Reprints of Papers on Electrostatics and Magnetism*, 2nd ed. (Macmillan, 1884), http://kirkmcd.princeton.edu/examples/EM/thomson_electrostatics_magnetism.pdf
(Macmillan, 1872), http://kirkmcd.princeton.edu/examples/EM/thomson_electrostatics_magnetism_72.pdf

[10] W. Thomson, *On the Theory of Magnetic Induction in Crystalline and Non-crystalline Substances*, Phil. Mag. **1**, 177 (1851), http://kirkmcd.princeton.edu/examples/EM/thomson_pm_1_177_51.pdf

[11] I. Newton, *Philosophiæ Naturalis Principia Mathematica* (1687), http://kirkmcd.princeton.edu/examples/mechanics/newton_principia.pdf

[12] W. Thomson, *On the Equations of Motion of Heat Referred to Curvilinear Co-ordinates*, Camb. Math. J. **4**, 33 (1843), http://kirkmcd.princeton.edu/examples/EM/thomson_cmj_4_33_43.pdf

[13] H.B. Dwight, *Tables of Integrals and Other Mathematical Data*, 4th ed. (Macmillan, 1961), http://kirkmcd.princeton.edu/examples/EM/dwight_57.pdf

[14] R. Arens, *Newton's observations about the field of a uniform thin spherical shell*, Note di Matematica **10** (Suppl. 1), 39 (1990),
http://kirkmcd.princeton.edu/examples/mechanics/arens_nm_10_s1_39_90.pdf

[15] M. Beleggia and M. De Graef, *General magnetostatic shape-shape interactions*, J. Magn. Magn. Mater. **285**, L1 (2005), http://kirkmcd.princeton.edu/examples/EM/beleggia_jmmm_285_L1_05.pdf