Small Oscillations of a Suspended Hoop

Alexander J. McDonald and Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(October 1, 2001)

1 Problem

Deduce the frequency of small oscillations of a hoop that is suspended from three strings of length l such that the plane of the hoop is always horizontal.

2 Solution

This problem illustrates the importance of choosing the appropriate coordinate for the analysis.

The hoop is constrained by the three strings such that if its center of mass moves vertically, the hoop rotates about a vertical axis through the cm. There is only one degree of freedom, and candidate coordinates are y, the height of the cm above its equilibrium position, and θ, the angle of rotation of the hoop about the axis through the cm. (Alternatively, the angular coordinate could be chosen as the angle between one of the support strings and the vertical.)

The kinetic energy is,

$$T = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}mr^2\dot{\theta}^2, \tag{1}$$

and the potential energy is,

$$V = mgy, \tag{2}$$

where m and r are the mass and the radius of the hoop, respectively.

The constrained motion relates coordinates y and θ as follows. When the hoop has rotated a small angle θ, the point of attachment to the hoop of one of the support strings has moved distance y vertically and distance $r\theta$ horizontally. Since the length of the support string is l, the vertical distance h from the support plane to the hoop is now,

$$h = \sqrt{l^2 - (r\theta)^2} \approx l - \frac{r^2\theta^2}{2l}\,. \tag{3}$$

During this motion, the cm of the hoop has risen by,

$$y = l - h \approx \frac{r^2\theta^2}{2l}\,. \tag{4}$$

Using this constraint relation we can now express the kinetic and potential energies in terms of a single coordinate, either y or θ.

It is tempting to use y as the favored coordinate, but this is a bad choice for a problem of small oscillations, because the potential energy (2) is not a quadratic function of y. Rather, we see that the potential energy is a quadratic function of θ, which will lead to a simple expression for oscillatory motion of this coordinate.

Hence, we write the kinetic and potential energies as,

$$T = \frac{1}{2}m\frac{r^4}{l^2}\theta^2\dot{\theta}^2 + \frac{1}{2}mr^2\dot{\theta}^2, \tag{5}$$

and ,

$$V = \frac{mgr^2\theta^2}{2l}, \tag{6}$$

noting from eq. (4) that,

$$\dot{y} = \frac{r^2\theta\dot{\theta}}{l}. \tag{7}$$

As we are concerned with small oscillations, we can ignore the first term in the kinetic energy (5) since it is small compared to the second, and we write,

$$T \approx \frac{1}{2}mr^2\dot{\theta}^2, \tag{8}$$

The Lagrangian (for small oscillations) is therefore,

$$L = T - V \approx \frac{1}{2}mr^2\left(\dot{\theta}^2 - \frac{g}{l}\theta^2\right), \tag{9}$$

which leads to the equation of motion,

$$\ddot{\theta} \approx -\frac{g}{l}\theta. \tag{10}$$

Hence, the angular frequency ω of small oscillations is,

$$\omega = \sqrt{\frac{g}{l}}, \tag{11}$$

which is the same as that of a simple pendulum of length l.

Suppose, however, we had chosen y rather than θ as the independent coordinate. Then, eqs. (4) and (7) tell us that,

$$\dot{\theta} = \sqrt{\frac{l}{2y}}\frac{\dot{y}}{r}, \tag{12}$$

and the kinetic energy is,

$$T = \frac{1}{2}m\dot{y}^2\left(1 + \frac{l}{2y}\right) \approx \frac{1}{2}m\frac{l\dot{y}^2}{2y}, \tag{13}$$

where the approximation holds for small y. This form is undefined at the equilibrium value of $y = 0$, which should be a warning that y was not a good choice of coordinate. Nonetheless, we might persist to consider the Lagrangian,

$$L = T - V \approx \frac{1}{2}m\left(\frac{l\dot{y}^2}{2y} - 2gy\right), \tag{14}$$

which leads to the equation of motion,

$$\ddot{y} - \frac{\dot{y}^2}{2y} \approx -2\frac{g}{l}y, \tag{15}$$

This is not an equation for oscillatory motion about the equilibrium value $y = 0$.

We recall that coordinate y takes on only positive values during the motion, so the constraint relation (4) suggests that we try a solution of the form,

$$y = y_0 \cos^2 \omega t, \tag{16}$$

for which,

$$\dot{y} = -2\omega y_0 \sin \omega t \cos \omega t, \tag{17}$$

and,

$$\ddot{y} = -2\omega^2 y_0 (\cos^2 \omega t - \sin^2 \omega t). \tag{18}$$

Substituting eqs. (16)-(18) into the equation of motion (15) we find the frequency ω to be given by eq. (11). However, without the insight that small oscillations about equilibrium occur in coordinate θ rather than in y, we would be stuck at eq. (15) – or if we erroneously ignored the term $\dot{y}^2/2y$ we would have incorrectly concluded that $\omega = \sqrt{2g/l}$.