Low-Frequency Electromagnetic Waves on a Twisted-Pair Transmission Line

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1 Problem

Discuss the electromagnetic waves that can propagate in the space around a transmission line whose form is a double helix of radius \( a \) and longitudinal period \( p \approx a \). The pitch angle \( \psi \) of the helical windings with respect to the transverse planes is given by,

\[
\cot \psi = k_p a = \frac{2\pi a}{p}.
\]  

(1)

The angle \( \theta \) of the windings with respect to the axis of the line is then \( \theta = \pi/2 - \psi \), i.e.,

\[
\tan \theta = k_p a.
\]  

(2)

Such lines are extensively used for telephone communication at low frequencies for which \( ka, kp \ll 1 \), where \( k = 2\pi/\lambda = \omega/v \) is the wave number at angular frequency \( \omega \), and \( v \) is the wave velocity. For the case that \( ka, kp \gg 1 \) the waves can be thought of following the helical conductors such that the group velocity along the axis of the helix is,

\[
v_{g,z} \approx c \cos \theta.
\]  

(3)

Show that even at low frequencies eq. (3) is a reasonable approximation when \( a \approx p \), but when \( a \ll p \) (a gentle twist) then \( v_{g,z} \approx c\sqrt{\cos \theta} \).
2 Solution

Despite the common use of twisted-pair transmission lines, this problem seems little discussed in the literature. In the case of two-dimensional conductors there exist transverse electromagnetic (TEM) waves of the form $e^{i(kz-\omega t)}$ times the (transverse) static electric and magnetic field patterns. However, TEM waves will not propagate along a twisted pair of wires, whose structure is three-dimensional.

Waves on a single helical conductor have been discussed in the context of traveling-wave amplifiers in the “sheath” approximation [1, 2], where only the part of the waves that are independent of azimuth are analyzed. A fairly general discussions of waves on twisted-pair conductors for $ka \approx kp \approx 1$ has been given in [3], again in the context of traveling-wave amplifiers.\footnote{See [4] for the case of cross-wound helices.}

Here, we emphasize the low-frequency behavior, when $ka, kp \ll 1$.

2.1 General Form of the Fields in Cylindrical Coordinates

We use a cylindrical coordinate system $(r, \phi, z)$ whose axis is that of the transmission line. We ignore the insulation typically found on the wires of a twisted-pair line, and assume that the space outside the wires is vacuum.

The electromagnetic fields $E$ and $B$ with time dependence $e^{-i\omega t}$ satisfy the vector Helmholtz equation,

$$(\nabla^2 + k_f^2)E, B = 0,$$ \hspace{1cm} (4)

outside the wires, where,

$$k_f = \frac{\omega}{c} = \frac{2\pi}{\lambda_f}. \hspace{1cm} (5)$$

However, in cylindrical coordinates only their $z$-components satisfy the scalar Helmholtz equation,

$$(\nabla^2 + k_f^2)E_z, B_z = 0.$$ \hspace{1cm} (6)

We look for wavefunctions for $E_z$ and $B_z$ that propagate in the $z$-direction with the form,

$$f_m(r) e^{-im\phi} e^{i(k_mz - \omega t)}, \hspace{1cm} (7)$$

where $m$ is an integer. The (right-handed) helical conductor rotates by $\phi = kp z = 2\pi z/p$ as $z$ increases, so we expect the wavefunction (7) to include this symmetry via a phase factor $e^{-im(\phi-kpz)}$ such that the waveform rotates as it advances. The $z$-dependent part of this phase contributes to the wave number $k_m$, which takes the form,\footnote{The magnetic fields of twisted pairs have been discussed in [5, 6, 7, 8]. Twisted-pair structures with large currents are used as undulators to generate energetic photon beams at particle accelerators (see, for example, [9]).}

$$k_m = k_0(\omega) + mk_p. \hspace{1cm} (8)$$

\footnote{The present case contrasts with that of so-called Bessel beams of order $m$ (see, for example, the Appendix of [10]) where the drive currents are limited to a small region in $z$, rather than being periodic in $z$, such that $k_m = k_0$ for any index $m$.}
We are mainly interested in waves that propagate in the $+z$ direction, for which the index $m$ must be non-negative at low frequencies where $0 < k_0 \ll k_p$.\(^4\)

The phase $\varphi_m$ of the wave function (7) is $\varphi_m = \mathbf{k}^{(m)} \cdot \mathbf{x} - \omega t = k_m z - m \phi - \omega t$, where the wave vector $\mathbf{k}^{(m)}$ is given by,

$$\mathbf{k}^{(m)} = \nabla \varphi_m = k_m \hat{z} - \frac{m}{r} \hat{\phi}.$$ \hspace{1cm} (9)

The phase velocity $v_{p,m}$ of a partial wave of index $m$ is,

$$v_{p,m} = \frac{\omega}{k^{(m)}} \mathbf{k}^{(m)} = \frac{ck_f}{k_m^2 + m^2/r^2} \left( k_m \hat{z} - \frac{m}{r} \hat{\phi} \right).$$ \hspace{1cm} (10)

We expect that $k_0 \lesssim k_f (\ll k_p)$ so that $v_{p,0} \lesssim c \hat{z}$, but for nonzero index $m$ we have that $k_m \approx mk_p$, and hence,

$$v_{p,m} \approx \frac{ck_f r}{m[1 + (k_p r)^2]} \left( k_p r \hat{z} - \hat{\phi} \right),$$ \hspace{1cm} (11)

which is small compared to $c$ at any value of $r$. The wave vector $\mathbf{k}^{(m)}$ (and the phase velocity $v_{p,m}$) make angle $\theta_k$ to the $z$-axis given by,

$$\tan \theta_k = -\frac{1}{k_p r}$$ \hspace{1cm} (12)

for any nonzero index $m$. Note that at $r = a$ the wave vector is at right angles to the direction of the helical windings, for which $\tan \theta = k_p a$.

The group velocity of a partial wave is,\(^5\)

$$v_{g,m} = \nabla_{\mathbf{k}^{(m)}} = \frac{\partial \omega}{\partial \mathbf{k}^{(m)}},$$ \hspace{1cm} (13)

whose only nonzero component is,

$$v_{g,m,z} = \frac{d \omega}{dk_m} = \frac{d \omega}{dk_m} \approx \frac{1}{dk_m/d\omega} = \frac{1}{dk_0/d\omega} = v_{g,0,z} \equiv v_{g,z},$$ \hspace{1cm} (14)

independent of index $m$. We expect that $v_{g,z} \lesssim c$ in the low-frequency limit.

Using eqs. (7)-(8) in the Helmholtz equation (6), we see that the radial function $f_m$ obeys the Bessel equation,

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{df_m}{dr} \right) - \left( k_m^2 - k_f^2 + \frac{m^2}{r^2} \right) f = 0,$$ \hspace{1cm} (15)

where $|k_m| \geq k_0 > k_f$. The solutions to eq. (15) should remain finite at $r = 0$ and $\infty$, so for $r < a$ we use the modified Bessel function $I_m(k'_m r)$, and for $r > a$ we use $K_m(k'_m r)$, where,

$$k'_m = \sqrt{k_m^2 - k_f^2}.$$ \hspace{1cm} (16)

\(^4\)Waves with index $m$ negative (both for single helix and double-helix configurations) have their phase and group velocities in opposite directions. An application of such waves is the **backward wave oscillator**. See, for example, [11].

\(^5\)See, for example, sec. 2.1 of [12].
That is, the longitudinal components of the electric and magnetic fields outside the wires have the forms,

\[
E_z(r < a) = \sum_m E_m \frac{I_m(k'_m r)}{I_m(k'_m a)} e^{-im\phi} e^{i(k_m z - \omega t)}, \quad E_z(r > a) = \sum_m E_m \frac{K_m(k'_m r)}{K_m(k'_m a)} e^{-im\phi} e^{i(k_m z - \omega t)},
\]

\[
B_z(r < a) = \sum_m B_m \frac{I_m(k'_m r)}{I_m(k'_m a)} e^{-im\phi} e^{i(k_m z - \omega t)}, \quad B_z(r > a) = \sum_m B_m \frac{K_m(k'_m r)}{K_m(k'_m a)} e^{-im\phi} e^{i(k_m z - \omega t)},
\]

where \(B_m\) and \(E_m\) are constants to be determined, and \(I_m(k'_m a) = dI_m(k'_m a)/dr\). In eq. (17) we have noted that the Maxwell equation \(\nabla \times \mathbf{E} = i\kappa_f \mathbf{B}\) (in Gaussian units) implies that \(E_z\) and \(E_\phi\) is continuous across the surface \(r = a\). We verify later that the normalization of coefficients \(B_m\) to \(I_m(k'_m a)\) and \(K_m(k'_m a)\) insures continuity of the magnetic field component \(B_z\) across this surface, as required by the Maxwell equation \(\nabla \cdot \mathbf{B} = 0\).

The waves are driven by the current density \(\mathbf{J}\) in the twisted pair, which we can write as,

\[
\mathbf{J}(x, t) = J(\phi, z, t)\delta(r - a)(\sin \theta \hat{\phi} + \cos \theta \hat{z}),
\]

which points along the local direction of the twisted-pair conductors, and is confined to a thin cylinder of radius \(a\). The wavefunction \(J(\phi, z, t)\) must have the same dependence on \(\phi, z\) and \(t\) as eqs. (17)-(18), namely,

\[
J(\phi, z, t) = \sum_m J_m e^{-im\phi} e^{i(k_m z - \omega t)},
\]

assuming that the current only flows in the direction of the helical windings.

For a twisted pair, the current at fixed \(z\) and azimuth \(\phi + \pi\) is opposite to that at azimuth \(\phi\), which implies that \(J_m\) is nonzero only for odd \(m\).

In the case of a pair of wires of small diameter, the expansion (20) has contributions from all odd integers \(m\). We will make a simplifying assumption that only the term \(m = 1\) is important, which corresponds to replacing the helical wires by a pair of helical wire bundles, each of which extends over \(\Delta \phi = \pi\), such that the current in the bundles at fixed \(z\) varies as \(\cos \phi\). If the peak current in each wire is \(I\), then,

\[
J(\phi, z, t) = \frac{I}{2a \cos \theta} e^{-i\phi} e^{i(k_1 z - \omega t)},
\]

\[
E_z(r < a) = E_1 \frac{I_1(k'_1 r)}{I_1(k'_1 a)} e^{-i\phi} e^{i(k_1 z - \omega t)}, \quad E_z(r > a) = E_1 \frac{K_1(k'_1 r)}{K_1(k'_1 a)} e^{-i\phi} e^{i(k_1 z - \omega t)},
\]

and

\[
B_z(r < a) = B_1 \frac{I_1(k'_1 r)}{I_1(k'_1 a)} e^{-i\phi} e^{i(k_1 z - \omega t)}, \quad B_z(r > a) = B_1 \frac{K_1(k'_1 r)}{K_1(k'_1 a)} e^{-i\phi} e^{i(k_1 z - \omega t)}.
\]

To deduce the other field components from the forms (17)-(18) it is useful to note that the electromagnetic fields can also be derived from the electric and magnetic Hertz vectors \(\mathbf{Z}_E\) and \(\mathbf{Z}_M\) (also called polarization potentials; see, for example, sec. 1.11 and chap. 6 of [13]),
each of which has only a \( z \)-component. These Hertz scalars, which we call \( Z_E \) and \( Z_M \), obey the scalar Helmholtz equation, \((\nabla^2 + k_i^2)Z_E, Z_M = 0\), outside the wires. Thus, the Hertz scalars also have the forms (22)-(23), and we will verify that,

\[
Z_E = -\frac{E_z}{k_1^2}, \quad Z_M = -\frac{B_z}{k_1^2}.
\]

(24)

The scalar and vector potentials \( V \) and \( \mathbf{A} \) are related to the Hertz vectors according to,

\[
V = -\nabla \cdot Z_E, \quad \mathbf{A} = \frac{1}{c} \frac{\partial Z_E}{\partial t} + \nabla \times Z_M,
\]

(25)

and hence the electric and magnetic fields \( \mathbf{E} \) and \( \mathbf{H} \) are given by,

\[
\mathbf{E} = \nabla(\nabla \cdot Z_E) - \frac{1}{c^2} \frac{\partial^2 Z_E}{\partial t^2} - \frac{1}{c} \nabla \times \frac{\partial Z_M}{\partial t}, \quad \mathbf{B} = \frac{1}{c} \nabla \times \frac{\partial Z_E}{\partial t} + \nabla \times (\nabla \times Z_M).
\]

(26)

The components of the electromagnetic fields in cylindrical coordinates in terms of the Hertz scalars \( Z_E \) and \( Z_M \) are (see sec. 6.1 of [13] with \( u^1 = r, u^2 = \phi, h_1 = 1 \) and \( h_2 = r \)),

\[
E_r = \frac{\partial^2 Z_E}{\partial r \partial z} - \frac{1}{cr} \frac{\partial Z_M}{\partial \phi \partial t}, \quad E_\phi = \frac{1}{r} \frac{\partial^2 Z_E}{\partial \phi \partial z} + \frac{1}{c} \frac{\partial Z_M}{\partial r \partial t}, \quad E_z = -\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( \frac{\partial Z_E}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \phi} \left( \frac{\partial Z_E}{\partial \phi} \right) \right],
\]

\[
B_r = \frac{\partial^2 Z_M}{\partial r \partial z} + \frac{1}{cr} \frac{\partial Z_E}{\partial \phi \partial t}, \quad B_\phi = \frac{1}{r} \frac{\partial^2 Z_M}{\partial \phi \partial z} - \frac{1}{c} \frac{\partial Z_E}{\partial r \partial t}, \quad B_z = -\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( \frac{\partial Z_M}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \phi} \left( \frac{\partial Z_M}{\partial \phi} \right) \right].
\]

(27)-(32)

For what it’s worth, the fields associated with \( Z_E \) are transverse magnetic (TM), while those associated with \( Z_M \) are transverse electric (TE).

To use the forms (22)-(23) in eqs. (27)-(32), we note that,

\[
I'_m(k'_m r) = k'_m I_{m-1} - \frac{m I_m}{r} = k'_m I_{m+1} + \frac{m I_m}{r}, \quad \frac{1}{r} \frac{d[r I'_m(k'_m r)]}{dr} = \left( k'_m^2 + \frac{m^2}{r} \right) I_m,
\]

\[
K'_m(k'_m r) = -k'_m K_{m-1} - \frac{m K_m}{r} = -k'_m K_{m+1} + \frac{m K_m}{r}, \quad \frac{1}{r} \frac{d[r K'_m(k'_m r)]}{dr} = \left( k'_m^2 + \frac{m^2}{r} \right) K_m,
\]

(33)-(34)

so that for \( r < a \) the field components are,

\[
E_r = -\frac{1}{k_1^2} \left[ i k_1 E_1 I'_1(k'_1 r) I_1(k'_1 a) + k_1^2 r B_1 I'_1(k'_1 r) I_1(k'_1 a) \right] e^{-i\phi} e^{i(k_1 z - \omega t)},
\]

(35)
\[ E_\phi = -\frac{1}{k_1^2} \left[ \frac{k_1}{r} E_1 \frac{I_1(k'_1 r)}{I_1(k'_1 a)} - i k_f B_1 \frac{I'_1(k'_1 r)}{I'_1(k'_1 a)} \right] e^{-i \phi} e^{i(k_1 z - \omega t)}, \]  
\[ E_z = -k_1^2 Z_E = E_1 \frac{I_1(k'_1 r)}{I_1(k'_1 a)} e^{-i \phi} e^{i(k_1 z - \omega t)}, \]  
\[ B_r = \frac{1}{k_1^2} \left[ \frac{k_f}{r} E_1 \frac{I'_1(k'_1 r)}{I_1(k'_1 a)} - i k_1 B_1 \frac{I'_1(k'_1 r)}{I'_1(k'_1 a)} \right] e^{-i \phi} e^{i(k_1 z - \omega t)}, \]  
\[ B_\phi = -\frac{1}{k_1^2} \left[ i k_f E_1 \frac{I'_1(k'_1 r)}{I_1(k'_1 a)} + k_1 B_1 \frac{I_1(k'_1 r)}{I_1(k'_1 a)} \right] e^{-i \phi} e^{i(k_1 z - \omega t)}, \]  
\[ B_z = -k_1^2 Z_M = B_1 \frac{I_1(k'_1 r)}{I_1(k'_1 a)} e^{-i \phi} e^{i(k_1 z - \omega t)}, \]

and for \( r > a \) we have the forms (35)-(40) with the substitution \( I_1 \rightarrow K_1 \).

We now see that the continuity of \( E_\phi \) and \( B_r \) across the surface \( r = a \), as previously mentioned, is satisfied by the above forms.

### 2.2 Determination of \( k_0 \) and the Group and Signal Velocities

The current in the helical windings is assumed to flow only at angle \( \theta \) with respect to the \( z \)-axis, so that for good conductors the conductivity of the “wires” is “infinite” in this direction, and zero in the perpendicular directions. Hence, the electric field on the surface of the cylinder \( r = a \) must be perpendicular to the direction of the current, \textit{i.e.},

\[ E_\phi(r = a) = -\cot \theta E_z(r = a), \]  
and hence,

\[ \left( k_1^2 a \cot \theta - k_1 \right) E_1 + i k_f a B_1 = 0. \]  

Also, the tangential component of the magnetic field in the direction of the current must be continuous at \( r = a \), which implies that,

\[ B_z(r = a_-) + \tan \theta B_\phi(r = a_-) = B_z(r = a_+) + \tan \theta B_\phi(r = a_+), \]  

and hence,

\[ i k_f a I'_1(k'_1 a) K_1'(k'_1 a) E_1 + \left( k_1^2 a \cot \theta - k_1 \right) I_1(k'_1 a) K_1(k'_1 a) B_1 = 0. \]  

For the simultaneous linear equations (42) and (44) to be consistent, the determinant of the coefficient matrix must vanish, \textit{i.e.},

\[ \left( k_1^2 a \cot \theta - k_1 \right)^2 = -(k_f a)^2 \frac{I'_1(k'_1 a) K_1'(k'_1 a)}{I_1(k'_1 a) K_1(k'_1 a)}. \]  

This determines \( k_0 \) (and therefore \( k_1 \) and \( k'_1 \)) in terms of \( a, p \) and \( k_f \).

We restrict our attention to low frequencies such that \( k_f a \ll 1 \). In the limit that \( k_f \) and \( k_0 \) vanish, then \( k_1 = k'_1 = k_p \) and \( k_1^2 a \cot \theta = k_p = 0 \), recalling that \( \cot \theta = 1/k_p a \), so that eq. (45) is satisfied. For small \( k_f \) and \( k_0 \) we approximate,

\[ k_1 = k_p + k_0 \approx k_p \left( 1 + \frac{k_0}{k_p} \right), \quad k_1^2 = k_1^2 - k_f^2 \approx k_p^2 \left( 1 + 2 \frac{k_0}{k_p} - \frac{k_f^2}{k_p^2} \right), \]
so that it suffices to take the arguments of the Bessel functions as \( k_p a \). Using these in eq. (45) and recalling eqs. (33)-(34), we find,

\[
\left( k_0 - \frac{k_p^2}{k_p^2} \right)^2 \approx k_0^2 \approx -(k_f a)^2 \frac{I_1'(k_p a) K_1'(k_p a)}{I_1(k_p a) K_1(k_p a)} = k_p^2 C^2(k_p a),
\]

where the constant \( C \) defined by,

\[
C^2(k_p a) = -a^2 \frac{I_1'(k_p a) K_1'(k_p a)}{I_1(k_p a) K_1(k_p a)} = \frac{[k_p a I_0(k_p a) - I_1(k_p a)][k_p a K_0(k_p a) + K_1(k_p a)]}{I_1(k_p a) K_1(k_p a)}
\]

is real and positive since \( K_1' \) is negative, as seen in the figure below.

\[
C^2(1) = \frac{1.2}{0.55} \approx 2.
\]

For example, if \( \theta = 45^\circ \) then \( k_p a = 1 \), and,

\[
C^2(1) \approx \frac{[1.2 - 0.55][0.4 + 0.6]}{0.55 \cdot 0.6} \approx 2,
\]

and \( C(1) \approx 1.4 \).

For \( k_p a \ll 1 \) (gentle twist) then \( I_0(k_p a) \approx 1 + (k_p a)^2 / 2, I_1(k_p a) \approx k_p a / 2 + (k_p a)^3 / 8 \), and \( K_1(k_p a) \gg k_p a K_0(k_p a) \), so we have,

\[
C^2(k_p a \ll 1) \approx \frac{k_p a I_0(k_p a)}{I_1(k_p a)} - 1 \approx 1 + (k_p a)^2 / 2 \approx \frac{1}{\cos \theta}.
\]

From eq. (47), the wave number \( k_0 \) is,

\[
k_0 \approx C k_f = C \frac{\omega}{c}.
\]
Recalling from eqs. (8)-(9) that \( k^{(1)} \equiv k = (k_0 + k_p) \hat{z} - \hat{\phi}/r \), eq. (51) can be recast as the dispersion relation,

\[
\omega = \omega(k^{(1)}) \equiv \omega(k) = \frac{c}{C} k_0 = \frac{c}{C} \left( k_z - \frac{k_p r}{r} \right) = \frac{c}{C} (k_z + k_p r k_\phi).
\]  

(52)

Then, the group velocity vector (13) is,

\[
v_g = \nabla_k \omega(k) = \frac{\partial \omega}{\partial k_z} \hat{z} + \frac{\partial \omega}{\partial k_\phi} \hat{\phi} \approx \frac{c}{C} (\hat{z} + k_p r \hat{\phi}).
\]

(53)

While the \( z \)-component, \( v_{g,z} \) of the group velocity is independent of radius \( r \), the group velocity vector \( v_g \) makes angle \( \theta_g \) to the \( z \)-axis given by,

\[
\tan \theta_g \approx k_p r.
\]

(54)

At very small \( r \) the group velocity is essentially parallel to the \( z \)-axis, but at large \( r \) lines of the group velocity form helices with very small pitch. The magnitude of the group velocity is,

\[
v_g \approx \frac{c}{C} \sqrt{1 + (k_p r)^2},
\]

(55)

which exceeds \( c \) at large \( r \). However, the signal velocity \( v_s \) is clearly,

\[
v_s = v_{g,z} = \frac{c}{C} < c.
\]

(56)

Comparing with eq. (12), we see that the group velocity \( v_g \) is perpendicular to the phase velocity \( v_p \), and that on the surface \( r = a \) the group velocity is along the direction of the helical windings.

For \( \theta = 45^\circ \) we find that \( v_{g,z} \approx c/C \approx 0.7 c \approx c \cos \theta \) for an uninsulated twisted-pair transmission line. This happens to be close to the group velocity of typical insulated, untwisted two-wire transmission lines!

For gently twisted, uninsulated pairs and low frequencies, eqs. (50) and (53) indicate that \( v_{g,z} \approx c \sqrt{\cos \theta} \).

### 2.3 Characteristic Impedance \( Z_0 \) at Low Frequencies

To evaluate the characteristic impedance of the transmission line at low frequencies, we consider the radial electric field (35) for \( r < a \), for which we need to know the constants \( B_1 \) and \( E_1 \) in terms of the (peak) current \( I \) in the windings.

We can relate \( B_1 \) to the (peak) current \( I \) in the twisted pair via Ampère’s law for a small loop of length \( dz \) in the \( r-z \) plane that surrounds a short segment of the conductor where the current is maximal,

\[
\frac{4\pi}{c} I_{\text{max}, \text{through loop}} = \frac{4\pi}{c} \int I \approx |B_z(r = a_-) - B_z(r = a_+)| dz
\]

\[
\approx B_1 \left( \frac{I_1(k_p a)}{I_1(k_p a)} - \frac{K_1(k_p a)}{K_1(k_p a)} \right) dz.
\]

(57)

The group velocity vector follows straight lines in homogenous media (see, for example, sec. 2.1 of [12]). Because of the twisted conductors, the present problem is not one of a homogenous medium, and the group velocity vector field need not have straight streamlines.
That is,
\[ B_1 = \frac{4\pi}{c} \frac{-I'_1(k_p a)K'_1(k_p a)}{p I'_1(k_p a)K_1(k_p a) - I_1(k_p a)K'_1(k_p a)} \approx \frac{2a}{I} = \frac{4\pi}{c} k_p C^2 D I, \] (58)

where,
\[ D(k_p a) = \frac{1}{a} \frac{I_1(k_p a)K_1(k_p a)}{I'_1(k_p a)K_1(k_p a) - I_1(k_p a)K'_1(k_p a)} \]
\[ = \frac{I_1(k_p a)K_1(k_p a)}{[k_p a I_0(k_p a) - I_1(k_p a)]K_1(k_p a) + I_1(k_p a)K_0(k_p a) + K_1(k_p a)}. \] (59)

Then, eqs. (42) and (51) tell us that,
\[ E_1 \approx -\frac{ik_f a}{k_0} B_1 \approx -\frac{ia}{C} B_1 = -\frac{4\pi i k_p}{c} C D I. \] (60)

From eq. (35) we see that the radial electric field for \( r < a \) is largely due to the term in \( E_1 \) since \( k_f \ll k_1 \) (at low frequencies). That is,
\[ E_r(r < a) \approx -\frac{i}{k_p} E_1 \frac{I_1(k_p r)}{I_1(k_p a)} e^{-i\phi} e^{i(k_1 z - \omega t)} = -\frac{4\pi}{c} C D I \frac{I'_1(k_p r)}{I_1(k_p a)} e^{-i\phi} e^{i(k_1 z - \omega t)}. \] (61)

The peak voltage difference between the opposing currents is therefore,
\[ V = 2 \int_0^a |E_r| \, dr \approx \frac{4\pi}{c} C D I = Z_0 I, \] (62)

where,
\[ Z_0 \approx 377 C D \Omega. \] (63)

When \( \theta = 45^\circ \),
\[ D \approx \frac{0.55 \cdot 0.6}{(1.2 - 0.44) \cdot 0.6 + 0.55 \cdot (0.4 + 0.6)} = 0.35, \] (64)
so that,
\[ Z_0(\theta = 45^\circ) \approx 377 \cdot 1.4 \cdot 0.35 = 185 \Omega. \] (65)

In practice, the wires of the twisted pair are insulated, which reduces the characteristic impedance to \( \approx 100 \Omega \).

For gentle twists \( (k_p a \ll 1) \) eq. (59) simplifies to,
\[ D \approx \frac{I_1(k_p a)}{k_p a I_0(k_p a)} \approx \frac{1}{2}, \] (66)
so that, recalling eq. (50),
\[ Z_0(\theta \approx 0) \approx \frac{189}{\sqrt{\cos \theta}} \Omega, \] (67)
little different from the value at \( \theta = 45^\circ \).
2.4 Energy Flux, Momentum and Angular Momentum Density

At low frequencies where \( k_1' \approx k_1 \approx k_p \gg k_f \) the electromagnetic fields for \( r < a \) follow from eq. (35)-(40) using eqs. (58) and (60) for the constants \( E_1 \) and \( B_1 \) in terms of the peak current \( I \),

\[
E_r \approx -\frac{4\pi}{c} \frac{CDI}{2} \frac{I_1'(k_pr)}{I_1(k_p a)} e^{-i\phi} e^{i(k_p z - \omega t)}, \tag{68}
\]

\[
E_\phi \approx \frac{4\pi}{c} iCDI \frac{I_1(k_pr)}{2r I_1'(k_p a)} e^{-i\phi} e^{i(k_p z - \omega t)}, \tag{69}
\]

\[
E_z \approx -\frac{4\pi}{c} \frac{k_p CDI}{2} \frac{I_1(k_pr)}{I_1(k_p a)} e^{-i\phi} e^{i(k_p z - \omega t)}, \tag{70}
\]

\[
B_r \approx -\frac{4\pi}{c} iC^2 DI \frac{I_1'(k_pr)}{2a} e^{-i\phi} e^{i(k_p z - \omega t)}, \tag{71}
\]

\[
B_\phi \approx -\frac{4\pi}{c} iC^2 DI \frac{I_1(k_pr)}{2ar} e^{-i\phi} e^{i(k_p z - \omega t)}, \tag{72}
\]

\[
B_z \approx \frac{4\pi}{c} k_p C^2 DI \frac{I_1(k_pr)}{2a} e^{-i\phi} e^{i(k_p z - \omega t)}, \tag{73}
\]

and for \( r > a \) we have the forms (68)-(73) with the substitution \( I_1 \rightarrow K_1 \).

The electric field components (68)-(70) have similar strength (in Gaussian units) to the magnetic field components (71)-(73). The latter correspond to the \( m = 1 \) term in the series expansions for the quasistatic magnetic fields given in [5]-[8].

The time-average Poynting vector \( \langle S \rangle \) for \( r < a \) at low frequencies is,

\[
\langle S \rangle = \frac{c}{8\pi} Re(\mathbf{E} \times \mathbf{B}^*) = \frac{c}{8\pi} Re[(E_\phi B^*_z - E_z B^*_\phi) \hat{r} + (E_z B^*_r - E_r B^*_z) \hat{\phi} + (E_r B^*_\phi - E_\phi B^*_r) \hat{z}] 
\approx \frac{4\pi}{c} \frac{C^3 D^2 I^2}{4a} \frac{I_1(k_pr)}{I_1'(k_p a)} \frac{I_1'(k_p a)}{I_1(k_p a)} \left[ k_p \hat{\phi} + \frac{\hat{z}}{r} \right], \tag{74}
\]

and that for \( r > a \) is obtained from eq. (74) with the substitution \( I_1 \rightarrow K_1 \).

At low frequencies there is no time-average flow of energy in the radial direction, and hence no radiation is emitted by the transmission line.\(^7\)

The energy-flow/Poynting vector (74) is in the same direction as the group velocity (53), as generally expected.\(^8\) Lines of the Poynting flux \( \langle S \rangle \) on the cylinder of radius \( r \) follow helices that make angle

\[
\theta_g \approx \tan^{-1} k_p r \tag{54}
\]

to the \( z \)-axis, such that only at \( r = a \) does the energy flow in a helix whose angle matches that of the windings, \( \theta \). At small \( r \) the (small) energy flows largely parallel to the axis. At large \( r \) the angle \( \theta_g \) approaches 90° and the Poynting vector is almost entirely transverse; however because \( K_1(k_pr) \rightarrow 0 \) at large \( r \) there is very little energy associated with these very tight spirals.

\(^7\)Even if we keep the smaller terms in \( E_\phi \) and \( B_\phi \) of eqs. (36) and (39) there is still no radiation emitted by the transmission line at low frequencies.

\(^8\)See, for example, sec. 2.1 of [12] and references therein.
The Poynting vector is at right angles to the wave vector \( \mathbf{9} \), whose angle \( \theta_k \) to the \( z \)-axis is given by eq. (12).

The Poynting vector plays the dual role of describing energy flux and momentum density, where the latter is given by,

\[
\langle \mathbf{p} \rangle = \frac{\langle \mathbf{S} \rangle}{c^2}
\]

in vacuum. The density \( \mathbf{1} \) of angular momentum in the electromagnetic field is therefore,

\[
\langle \mathbf{1} \rangle = \mathbf{r} \times \langle \mathbf{p} \rangle = \mathbf{r} \times \frac{\langle \mathbf{S} \rangle}{c^2}.
\]

On averaging over azimuth \( \phi \) only the \( z \)-component of the angular momentum is nonzero,

\[
\langle \mathbf{1} \rangle = \frac{4\pi}{c} \frac{C^3 D^2 I^2 I_1(k_\rho r) I_1'(k_\rho r)}{4a} \frac{k_\rho r}{I_1(k_\rho a) I_1'(k_\rho a)} \frac{\hbar}{c^2} \mathbf{\hat{z}}.
\]

Thus, the electromagnetic waves on a right-handed twisted-pair transmission line carry positive angular momentum. In a quantum view, the photons of the wave have angular momentum \( \hbar \) and energy \( \hbar \omega \). Hence, we expect that \( \langle \mathbf{1} \rangle = \langle \mathbf{u} \rangle / \omega \mathbf{\hat{z}} \) where \( \langle \mathbf{u} \rangle = (|\mathbf{E}|^2 + |\mathbf{B}|^2)/8\pi \) is the time-average electromagnetic energy density. However, this relation is not self evident given the description of the waves in terms of Bessel functions.

**A Appendix: A Single Wire Helix**

We can compare the twisted-pair transmission line to the case of a single helical wire \([1, 2]\) in the “sheath” approximation that the helical current flows at angle \( \psi \) uniformly over the entire cylinder \( r = a \), such that the current and fields have no azimuthal dependence. Then, instead of eqs. (35)-(40) \( r < a \), we now have,

\[
E_r = -\frac{ik_1}{k_0^2} E_0 \frac{I_0(k_0 r)}{I_0(k_0 a)} e^{i(k_0 z - \omega t)},
\]

\[
E_\phi = \frac{ik_f}{k_0^2} B_0 \frac{I_0'(k_0 r)}{I_0(k_0 a)} e^{i(k_0 z - \omega t)},
\]

\[
E_z = E_0 \frac{I_0(k_0 r)}{I_0(k_0 a)} e^{i(k_0 z - \omega t)},
\]

\[
B_r = -\frac{ik_1}{k_1^2} B_0 \frac{I_0'(k_0 r)}{I_0(k_0 a)} e^{i(k_0 z - \omega t)},
\]

\[
B_\phi = -\frac{ik_f}{k_1} E_0 \frac{I_0'(k_0 r)}{I_0(k_0 a)} e^{i(k_0 z - \omega t)},
\]

\[
B_z = B_0 \frac{I_0(k_0 r)}{I_0(k_0 a)} e^{i(k_0 z - \omega t)},
\]

and for \( r > a \) we have the forms (78)-(83) with the substitution \( I_0 \to K_0 \).

The condition (41) now implies that,

\[
k_0^2 E_0 + ik_f \cot \psi B_0 = 0.
\]
Similarly, the condition (43) implies that,

\[ ik_f \cot \psi I'_0(k'_0 a)K'_0(k'_0 a)E_0 + k'_0^2 I_0(k'_0 a)K_0(k'_0 a)B_0 = 0. \]  \(85\)

The vanishing of the determinant of the coefficient matrix tells us that,

\[ k'_0^4 = -k_f^2 \cot^2 \psi \frac{I'_0(k'_0 a)K'_0(k'_0 a)}{I_0(k'_0 a)K_0(k'_0 a)} = k'_0^2 k_f^2 \cot^2 \psi \frac{I_1(k'_0 a)K_1(k'_0 a)}{I_0(k'_0 a)K_0(k'_0 a)}, \]  \(86\)

recalling eqs. (33)-(34). That is,

\[ k'_0 \sqrt{\frac{I_0(k'_0 a)K_0(k'_0 a)}{I_1(k'_0 a)K_1(k'_0 a)}} = k_f \cot \psi. \]  \(87\)

At low frequencies such that \(k_f a \ll 1\) the factor involving Bessel functions in eq. (87) becomes large, and \(k'_0 \ll k_f\), as illustrated in the figure below, from [1].

![Diagram](kirkmcd.princeton.edu/examples/EM/chodorow_jap_26_33_55.pdf)

Then, \(k_0 = \sqrt{k_f^2 + k'_0^2} \approx k_f\) so that the phase velocity and group velocity are both very close to \(c\).

References

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