1 Problem

A particle of charge \( e \) with velocity \( \mathbf{v} = v \hat{z} \) passes through a metallic beam window at \( z = 0 \) and emerges into vacuum for \( z > 0 \). What is the frequency-angle spectrum of the radiation in the region \( z > 0 \), assuming that the beam window is perfectly conducting and an infinite sheet?

Consider also a charge \( e \) that moves through two semi-infinite media, with dielectric constants \( \epsilon_1 \) for \( z < 0 \) and \( \epsilon_2 > \epsilon_1 \) for \( z > 0 \), on trajectory \( \mathbf{r} = vt \hat{z} \) where \( v < c/n_2 = c/\sqrt{\epsilon_2} < c/n_1 \), where \( c \) is the speed of light in vacuum, and \( n = \sqrt{\epsilon} \) is the index of refraction of a dielectric (that has unit relative permeability).

2 Solution

We consider a method [1, 2] that worked well in characterizing Čerenkov radiation, based on an expression for the spectrum of energy vs. angular frequency \( \omega \) and solid angle \( \Omega \) of a pulse of radiation due to electric charge \( e \) with (generally time-dependent) velocity \( \mathbf{v} \),

\[
\frac{dU_\omega}{d\Omega} = \frac{e^2}{4\pi^2c} \left| \int_{-\infty}^{\infty} \mathbf{n} \times (\mathbf{n} \times \mathbf{\beta}) e^{i(\omega t - k \cdot r)} dt \right|^2 = \frac{e^2 n}{4\pi^2c} \left| \int_{-\infty}^{\infty} \mathbf{\beta} \times \mathbf{k} e^{i(\omega t - k \cdot r)} dt \right|^2
\]  

in Gaussian units, and where \( \mathbf{k} \) is the wave vector with \( k = n \omega/c \) in case of a medium with index of refraction \( n \), and hence \( \mathbf{k} = \hat{\mathbf{n}} \) is the unit vector pointing to the observer. Also, \( \beta = v/c \).

2.1 Metal-Vacuum Interface

2.1.1 A First Approximation

We first apply eq. (1) simply to the motion of charge \( e \) with position \( \mathbf{r} = vt \hat{z} = \beta ct \hat{z} \) for constant velocity \( \mathbf{v} = v \hat{z} \) for \( z > 0 \) in vacuum, where \( \mathbf{k} = \omega \hat{\mathbf{k}}/c = \omega (\sin \theta, 0, \cos \theta)/c \) for an observer in the \( x-z \) plane,

\[
\frac{dU_\omega}{d\Omega} = \frac{\omega^2}{4\pi^2c} \left| \int_{0}^{\infty} \mathbf{\beta} \hat{\mathbf{z}} \times \mathbf{k} e^{i(\omega t - k \cdot r)} dt \right|^2 = \frac{e^2 \beta^2 \omega^2}{4\pi^2c} \left| \int_{0}^{\infty} \sin \theta e^{i\omega t(1 - \beta \cos \theta)} dt \right|^2 = \frac{e^2 \beta^2 \omega^2 \sin^2 \theta}{4\pi^2c(1 - \beta \cos \theta)^2}
\]

\[\text{(2)}\]
taking \( e^{i\pi} = 0 \) as representing the time-average of the oscillatory function, and using Dwight 90.2 and 92.2 [6]. In the relativistic limit, \( \beta \to 1 \), where most observations of transition radiation have been made,

\[
U_\omega \to \frac{e^2 \ln 2\gamma}{\pi c}, \quad \text{with} \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}.
\]

As \( \beta \to 0 \), eq. (3) goes to the nonzero value \( e^2/2\pi c \).

### 2.1.2 A Better Approximation – Image Method

The preceding analysis ignored the effect of the time-dependent charge density induced on the surface of the conducting sheet at \( z = 0 \). In a better approximation we suppose that this effect is equivalent to the presence of an image charge \(-e\) at \( z = -vt \) for \( t = 0 \) (and the conducting sheet is absent). For the actual situation, the surface charge at time \( t > 0 \) exists on the sheet only inside a circle of radius \( ct \) about the origin, which the image method implies that the surface charge density is nonzero everywhere on the sheet for \( t > 0 \), so the result of this section is not “exact”.

Adapting eq. (1) to include the image charge, we have,

\[
\frac{dU_\omega}{d\Omega} = \frac{\omega^2}{4\pi^2 c} \left| \int_0^\infty e^{i\omega t(1 - \beta \cos \theta)} dt + \int_0^\infty e^{i\omega t(1 + \beta \cos \theta)} dt \right|^2
= \frac{e^2 \omega^2 \beta^2 \sin^2 \theta}{4\pi^2 c} \left| \frac{e^{i\omega \infty(1 - \beta \cos \theta)} - 1}{i\omega(1 - \beta \cos \theta)} + \frac{e^{i\omega \infty(1 + \beta \cos \theta)} - 1}{i\omega(1 + \beta \cos \theta)} \right|^2
= \frac{e^2 \beta^2 \sin^2 \theta}{\pi^2 c} \frac{\sin^2 \theta}{1 - \beta^2 \cos^2 \theta},
\]

where \( \boldsymbol{k} = \omega \mathbf{k}/c = \omega(\sin \theta, 0, \cos \theta)/c \) is in the direction of the radiation to the observer (located at large \( z > 0 \)), \( \hat{n} = \hat{\mathbf{n}} \). \( \beta_{e,-e} = \pm \hat{v} \cdot \hat{z}/c = \pm \beta \hat{z} \), \( \mathbf{r}_{e,-e} = \pm vt \hat{z} = \pm \beta c t \hat{z} \), and we take \( e^{i\omega \infty(1 \pm \beta \cos \theta)} = 0 \), as representing the time-average of the oscillatory behavior at large times.

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The result (5) was first obtained in [7]. See also sec. 2.1.2 of [8].

This result was obtained by a different approximation, called “quasi-classical” in [9]. See also sec. 28b, particularly p. 283, of [10].
The frequency spectrum of the transition radiation is, noting that \( 0 < \theta < \pi / 2 \) for an observer with \( z > 0 \),

\[
U_\omega = \frac{dU_\omega}{d\Omega} = \frac{2e^2}{\pi c^3 \beta^2} \int_0^1 \frac{1 - \cos \theta}{(1/\beta^2 - \cos^2 \theta)^2} d\cos \theta
\]

\[
= \frac{2e^2}{\pi c^3 \beta^2} \left[ \frac{\beta^2 \cos \theta}{2(1/\beta^2 - \cos^2 \theta)} + \frac{\beta^3}{4} \ln \frac{1/\beta + \cos \theta}{1/\beta - \cos \theta} - \frac{\cos \theta}{2(1/\beta^2 - \cos^2 \theta)} + \frac{\beta}{4} \ln \frac{1/\beta + \cos \theta}{1/\beta - \cos \theta} \right]_0^1
\]

\[
= \frac{2e^2}{\pi c^3 \beta^2} \left[ -\frac{\beta^2}{2} + \frac{\beta(1 + \beta^2)}{4} \ln(\gamma^2(1 + \beta^2)) \right] = \frac{e^2}{\pi c} \left[ \frac{1 + \beta^2}{\beta} \ln \gamma(1 + \beta) - 1 \right],
\]

using Dwight 140.2 and 142.2 [6]. In the relativistic limit, \( \beta \to 1 \),

\[
U_\omega \to \frac{2e^2 \ln 2\gamma}{\pi c}, \tag{7}
\]

which is twice that found in the first approximation (4). As \( \beta \to 0 \), eq. (6) goes to zero.

### 2.2 Interface between Two Dielectrics

#### 2.2.1 A First Approximation

For a first approximation we apply a version of eq. (1) considering only the motion \( r = vt \hat{z} = \beta ct \hat{z} \) of the charge \( e \), assumed to constant velocity \( v = v \hat{z} \) both in the medium with (relative) dielectric constant \( \epsilon_1 \) at \( z < 0 \) and in the medium with (relative) dielectric constant \( \epsilon_2 \) at \( z > 0 \). We consider only the forward radiation \( |\theta| < \pi / 2 \) received by an observer in the \( x-z \) plane with large \( z > 0 \). If a ray observed at angle \( \theta_2 \) for \( z > 0 \) originated at \( z < 0 \) it had angle \( \theta_1 \) for \( z < 0 \) related by Snell’s law, \( n_1 \sin \theta_1 = n_2 \sin \theta_2 \), where we take the indices of refraction to be \( n_{1,2} = \sqrt{\epsilon_{1,2}} \). To avoid complications of total internal reflection at the interface of rays emanating from \( z < 0 \), we suppose that \( n_1 < n_2 \), i.e., \( \epsilon_1 < \epsilon_2 \). Then, noting that \( \mathbf{k}_i = n_i \omega \mathbf{k}_i / c \), and setting \( e^{\pm i\omega} = 0 \) as before,

\[
\frac{dU_\omega}{d\Omega} = \frac{n_2 \omega^2}{4\pi^2 c} \left| \int_{-\infty}^0 e^{i\mathbf{k}_1 \cdot \mathbf{r}} e^{i(\omega t - \mathbf{k}_1 \cdot \mathbf{r})} dt + \int_0^\infty e^{i\mathbf{k}_2 \cdot \mathbf{r}} e^{i(\omega t - \mathbf{k}_2 \cdot \mathbf{r})} dt \right|^2
\]

\[
= \frac{e^2 n_2 \omega^2 \beta^2}{4\pi^2 c} \left| \sin \theta_1 \int_{-\infty}^0 e^{i\omega(1-n_1 \beta \cos \theta_1)} dt + \sin \theta_2 \int_0^\infty e^{i\omega(1-n_2 \beta \cos \theta_2)} dt \right|^2
\]

\[
= \frac{e^2 n_2 \omega^2 \beta^2}{4\pi^2 c} \left| \frac{n_2}{n_1} \sin \theta_2 \int_{-\infty}^0 e^{i\omega(1-n_1 \beta \cos \theta_1)} dt + \sin \theta_1 \int_0^\infty e^{i\omega(1-n_2 \beta \cos \theta_2)} dt \right|^2
\]

\[
= \frac{e^2 n_2 \omega^2 \beta^2 \sin^2 \theta_2}{4\pi^2 c n_1^2} \left| \frac{n_2}{n_1} \frac{1 - e^{-i\omega\infty(1-n_1 \beta \cos \theta_1)}}{i\omega(1-n_1 \beta \cos \theta_1)} + \frac{n_1}{n_1} \frac{e^{i\omega\infty(1-n_2 \beta \cos \theta_2)} - 1}{i\omega(1-n_2 \beta \cos \theta_2)} \right|^2
\]

\[
= \frac{e^2 n_2 \omega^2 \beta^2 \sin^2 \theta_2}{4\pi^2 c n_1^2} \left| \frac{n_2}{1-n_1 \beta \cos \theta_1} - \frac{n_1}{1-n_2 \beta \cos \theta_2} \right|^2. \tag{8}
\]
The transition radiation of eq. (8) is large only when the denominators in the last line are small, *i.e.*, when \( n_i, \beta \) and \( \cos \theta_i \) are all close to 1. We recall the atomic model of the frequency dependence of the dielectric constant \( \epsilon \),

\[
\epsilon(\omega) = 1 + \frac{4\pi Ne^2}{m} \sum_j \frac{f_j}{\omega_j - \omega^2 - i\Gamma_j \omega}, \quad \sum_j f_j = 1,
\]

where \( N \) is the number density of atoms, \( e \) and \( m \) are the charge and mass of an electron, \( f_j \) is the relative strength (oscillator strength) of oscillation \( j \) in the atom, with angular frequency \( \omega_j \) and damping constant \( \Gamma_j \). The dielectric constant is near 1 only for high frequencies, in which case,

\[
\epsilon(\omega \text{ large}) \approx 1 - \frac{4\pi Ne^2}{m \omega^2} = 1 - \frac{\omega_p^2}{\omega^2},
\]

where \( \omega_p = \sqrt{4\pi Ne^2/m} \) is the plasma frequency of the medium. The corresponding index of refraction is,

\[
n(\omega \text{ large}) = \sqrt{\epsilon(\omega \text{ large})} \approx 1 - \frac{\omega_p^2}{2\omega^2}.
\]

For \( \beta \) near 1, we write \( \beta = \sqrt{1 - 1/\gamma^2} \approx 1 - 1/2\gamma^2 \), and of \( \cos \theta_i \) near 1 we have \( \cos \theta_i \approx 1 - \theta_i^2/2 \). In these limits,

\[
\frac{1}{1 - n_i \beta \cos \theta_i} \approx 1 - \frac{1}{(1 - \omega_{p_i}^2/2\omega^2)(1 - 1/2\gamma^2)(1 - \theta_i^2/2)} \approx \frac{2}{\omega_{p_i}^2/\omega^2 + 1/\gamma^2 + \theta_i^2},
\]

and the frequency-angle spectrum of the transition radiation is, noting that \( \theta_1 \approx \theta_2 \) since \( n_1 \approx n_2 \approx 1 \),

\[
\frac{dU}{d\Omega} \approx \frac{e^2 \theta_2^2}{\pi^2 c} \left[ \frac{1}{\omega_{p_1}^2/\omega^2 + 1/\gamma^2 + \theta_2^2} - \frac{1}{\omega_{p_2}^2/\omega^2 + 1/\gamma^2 + \theta_2^2} \right]^2.
\]

As expected, this vanishes if the two media are the same, *i.e.*, if \( \omega_{p_1} = \omega_{p_2} \).

The angular distribution peaks for,

\[
\frac{\omega_{p_1}^2}{\omega^2} + \frac{1}{\gamma^2} < \theta_2^2 < \frac{\omega_{p_2}^2}{\omega^2},
\]

*i.e.*, in the forward direction (like Bremsstrahlung, such that there was early skepticism that transition radiation is not distinct from Bremsstrahlung).

\footnote{See, for example, p. 135 of [11] or sec. 7.5 of [5]}
\footnote{At the high frequencies where the index of refraction \( n \) is less than 1, there is no Čerenkov radiation, so in practice the issue of interference between Čerenkov and transition radiation is moot.}
The angular distribution can be integrated to give, with \( x = \theta_2^2 \) and \( A_i = \omega_{p_i}^2/\omega^2 + 1/\gamma^2 \),

\[
U_\omega(\omega \text{ large}) = \int \frac{dU_\omega}{d\Omega} d\Omega \approx 2\pi \int_0^\infty \frac{dU_\omega}{d\Omega} \theta_2 d\theta_2 = \pi \int_0^\infty \frac{dU_\omega}{d\Omega} d\theta_2^2
\]

\[
\approx \frac{e^2}{\pi c} \int_0^\infty x \left| \frac{1}{A_1 + x} - \frac{1}{A_2 + x} \right|^2 dx \approx \frac{e^2(A_1 - A_2)^2}{\pi c} \left[ \ln \frac{A_1 + x}{A_2 + x} + \frac{1}{(A_1 - A_2)^2} \right]_{0}^{\infty} \left( \frac{A_1 + x}{A_2 + x} \right) = \left( \frac{A_1 + A_2}{(A_1 - A_2)^2} \right) \ln \frac{A_2}{A_1} - 2
\]

\[
= \frac{e^2}{\pi c} \left[ \frac{A_1 + A_2}{(A_1 - A_2)^2} \ln \frac{A_2}{A_1} - 2 \right] = \frac{e^2}{\pi c} \left[ \frac{\omega_{p_1}^2 + \omega_{p_2}^2 + 2\omega^2/\gamma^2}{\omega_{p_1}^2 - \omega_{p_2}^2} \right] \ln \left( \frac{\omega_{p_2}^2 + \omega^2/\gamma^2}{\omega_{p_1}^2 + \omega^2/\gamma^2} \right) - 2
\]

\[
\approx \frac{e^2 \gamma^4}{6\pi c \omega^4} (\omega_{p_1}^2 - \omega_{p_2}^2)^2.
\]  

(15)

using Dwight 113.1 [6], and where the last approximation requires evaluation to third order.

The photon number spectrum is obtained by dividing eq. (15) by \( h \),

\[
N_\omega = \frac{U_\omega}{\hbar} \approx \frac{1}{6\pi \hbar c \omega^4} (\omega_{p_1}^2 - \omega_{p_2}^2)^2 = \frac{1}{6\pi} \frac{1}{137} \frac{\gamma^4}{\omega^2} (\omega_{p_1}^2 - \omega_{p_2}^2)^2,
\]

which is a very weak effect, although it does vary as \( \gamma^4 \), so can be significant for ultrarelativistic particles.

Our eq. (8) appears to be rather different than the Ginzburg-Frank result,\(^5\)

\[
dU_\omega = \frac{e^2 v^2 n_2 \sin^2 \theta_2 \cos^2 \theta_2}{\pi^2 c^3} \left| \frac{\epsilon_1 - \epsilon_2}{(1 - \epsilon_2 \beta^2 \cos^2 \theta_2) \left( 1 - \beta \sqrt{\epsilon_1 - \epsilon_2 \sin^2 \theta_2} \right)} \right|^2 \left( 1 - \beta^2 \epsilon_2 - \beta \sqrt{\epsilon_1 - \epsilon_2 \sin^2 \theta_2} \right)^2 \left( \epsilon_1 \cos \theta_2 + \sqrt{\epsilon_1 \epsilon_2 - \epsilon_2 \sin^2 \theta_2} \right),
\]

(17)

but in the limit of high frequency and high velocity, eqs. (8) and (17) lead to the same results, our eqs. (13)-(16),\(^6\) so it is perhaps not necessary to seek better approximations than that of this section.

2.2.2 A Better Approximation – Image Method

The method of sec. 2.2.1 ignored the effects of time-dependent polarization charges near the interface \( z = 0 \) between the two semi-infinite dielectric media with (relative) dielectric constants \( \epsilon_1(z < 0) \) and \( \epsilon_2(z > 0) \). For a better approximation, we recall the image method for dielectrics,\(^7\) that when charge \( e \) is at \((0, 0, z)\) in medium 2, the electric field for \( z > 0 \)

\(^5\)See eq. (24.22) of [10], or eq. (2.41) of [8].

\(^6\)Compare with eq. (2.59), p. 35 of [8].

\(^7\)See, for example, sec. 2.1.1 of [12]. Conventions differ in the dielectric image method. Sec. 4.4 of [5] supposes that the image charge is not in vacuum, but in a medium with dielectric constant \( \epsilon_2 \), while sec. 5.05 of [13] supposes the image charge is in a medium of dielectric constant \( \epsilon_1 \).
is that in vacuum due to effective charge \( e/\epsilon_2 \) at \((0,0,z)\) and an image charge \(-e/\epsilon_2)((\epsilon_1 - \epsilon_2)/(\epsilon_1 + \epsilon_2))\), while the field for \( z < 0 \) is that due to effective charge \( 2e/(\epsilon_1 + \epsilon_2) \) at \((0,0,z)\) in vacuum.

We consider the case of an observer with \( z > 0 \) (i.e., forward radiation), such that a ray with angle \( \theta_2 \) to the \( z \)-axis at the observer, if it originates with \( z < 0 \), makes angle \( \theta_1 \) to the \( z \)-axis related by Snell’s law, \( n_1 \sin \theta_1 = n_2 \sin \theta_2 \), i.e., \( \sqrt{\epsilon_1} \sin \theta_1 = \sqrt{\epsilon_2} \sin \theta_2 \). Then, the frequency-angle spectrum of Čerenkov radiation by charge \( e \) with position \( \mathbf{x} = vt \hat{z} \) and \( v > c/n_{12} \) follows from eq. (1) as,\(^8\)

\[
\frac{dU_\omega}{d\Omega} = \frac{\omega^2 n_2}{4\pi^2 c^3} \left[ \int_0^\infty \frac{e}{\epsilon_2} \beta \times \hat{k}_2 e^{i\omega t(1-n_2 \beta \cos \theta_2)} \frac{1}{\epsilon_1 + \epsilon_2} \sin \theta_2 \frac{1}{i\omega(1-n_2 \beta \cos \theta_2)} d\theta_2 \right]
\]

This does vanish if \( \epsilon_1 = \epsilon_2 \), but is not quite the same as the Ginzburg-Frank result (17).

References


http://kirkmcd.princeton.edu/examples/cerenkov_n.pdf

http://kirkmcd.princeton.edu/examples/QM/schiff_qm_49.pdf


http://kirkmcd.princeton.edu/examples/EM/jackson_ce2_75.pdf

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\(^8\)The contribution from the image charge at \( z = -vt \) for \( t > 0 \), when charge \( e \) is at \( z = vt \), does not actually originate at \( z < 0 \), but rather at \( z = 0 \). As such, for radiation observed at \( z > 0 \) we use \( \hat{k}_2 \) rather than \( \hat{k}_1 \) in the second integral of the first line of eq. (18).


