Comments on Torque Analyses

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1 Introduction

While **Newton**'s laws of motion, p. 83 of [1], are generally sufficient to discuss the motion of a single, point mass, analysis of more complicated systems benefits from additional insights. Such systems often include "bodies" that can be approximated as "rigid", in which case a Newtonian analysis can/should include the concepts of torque and angular momentum, these being the "moments" of force and (linear) momentum about some point. In cases where a potential energy can be identified, one can instead follow **Lagrange** [2] to deduce equations of motion without consideration of forces (or torques).

In this note we emphasize "Newtonian" methods involving torque[1] and angular momentum[2] (although Newton himself never used these concepts[3]).

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[1]In the discussion of the motion of a balance with unequal weights by a follower of **Aristotle**, prob. 3, p. 353 of [3], and p. 10 of [4], it was argued that the ensuing rotary motion is related to the product of the weight times the distance from the fulcrum, which represents an early concept of torque. The Greek word for torque, $\rho o \pi \eta$, can also mean *weight*, and appears on p. 348 of [3] in that sense. See also [5].

The term quan in the Chinese **Mohist** Canon ($\approx$ 300 BC) [6] could be interpreted as meaning torque.

Euclid's *Book on the Balance* ($\approx$ 250 BC), known only from an Arabic copy, contains a term translated [7] as "force of weight", meaning torque.

Galileo introduced the term moment for the torque component about a fixed axis (pp. 8-9 of [8], in French, *momento* on p. 159 of [9]) as his translation of $\rho o \pi \eta$ (see also p. 120 of [10]).

This moment/torque has dimensions of work, and was previously identified by **Jordanus de Nemore** under the name *gravitas secundum situm* (positional weight) in arguments that were early applications of the principle of virtual work. [11] See also chap. 3 of [10], and [12].

Another term for torque is couple, popularized by **Poinsot**. See, for example, p. 14 of [13], and [14].

The term torque was introduced only in 1885 by **James Thomson** (brother of **Lord Kelvin**), President of the Institution of Engineers and Shipbuilders in Scotland, p. 91 of [15], in a comment on a paper on steam engines. This term was immediately adopted by **Thompson**, p. 388 of [16], which lists six alternatives.

[2]The concept of angular momentum is attributed by **Truesdell**, p. 252 of [17], to **James Bernoulli** in 1686 (before publication of Newton's *Principia*), while the first use of angular momentum in a mathematical analysis, reported on p. 256 of [17], is due to **David Bernoulli** (nephew of James) (1744), where it was called simply momentum.

The earliest recorded used of the term momentum to mean mass times velocity was on p. 67 of [18] (1721).

An early use of the term angular momentum is on p. 17 of [19] (1818), in a manner that indicates this was not the first such usage. **Rankine**, p. 506 of [20] (1858) attributes this term to **Hayward**, p. 7 of [21] (1856). This quantity is often called instead the moment of momentum, or momentum of rotation as on p. 313 of [22] (1785).

For additional historical commentary on angular momentum, see [23].

[3]Newton's 1st law reads (in translation, p. 83 of [1]): *Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.*

This appears to be equivalent to the "law" that the linear momentum of a body is constant if it is subject to no external force. However, the second sentence after the statement of the first law reads: *A top, whose parts by their cohesion are perpetually drawn aside from rectilinear motions, does not cease its rotation, otherwise than as it is retarded by the air.* Here it seems that Newton had some concept of angular momentum as

2 Newton's Equations of Motion according to Euler

Prior to Newton, the (vector) concepts of force (already present in the works of Aristotle) and *impetus* (due to **Buridan**, ≈ 1350 [25, 26], and which included both linear and angular momentum) were less successful as explanations of physical phenomena than use of the scalar quantities such as the moment of a weight.[4]

Newton expressed his 2nd law verbally (p. 823 of [1]): *The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.* This was first re-expressed as the now-familiar equation,

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt} m\,\mathbf{v} = m\,\mathbf{a} = \mathbf{F}, \tag{1}$$

for a particle of time-independent mass m at position $\mathbf{x}$ with velocity $\mathbf{v} = d\mathbf{x}/dt = \dot{\mathbf{x}}$, momentum $\mathbf{p} = m\,\mathbf{v}$, acceleration $\mathbf{a} = d\mathbf{v}/dt$, where $\mathbf{F}$ is the force on the particle, by **Euler** in 1744 [28].[5] Euler extended his discussion to include angular momentum, leading to a general analysis of the motion of a collection of points masses, and of a rigid body, first in [29] (1751), then in his long book [30] (1765), and most definitively in [31] (1775).

Euler introduced the angular momentum (moment of momentum) $\mathbf{L}$ with respect to the fixed origin of an inertial frame,[6]

$$\mathbf{L} = \mathbf{x} \times \mathbf{p} = \mathbf{x} \times m\,\mathbf{v}, \tag{2}$$

which for a time-independent mass obeys the relation,

$$\frac{d\mathbf{L}}{dt} = \mathbf{v} \times m\,\mathbf{v} + \mathbf{x} \times \frac{d\mathbf{p}}{dt} = \mathbf{x} \times \mathbf{F} \equiv \boldsymbol{\tau}, \tag{3}$$

where the torque $\boldsymbol{\tau}$ is defined with respect to the origin.

For a set of particles, labeled by subscript i, of time-independent masses Euler wrote,

$$m = \sum_i m_i, \quad m\mathbf{x}_{\rm cm} = \sum_i m_i \mathbf{x}_i, \quad m\mathbf{v}_{\rm cm} = \sum_i m_i \mathbf{v}_i, \quad m\,\mathbf{a}_{\rm cm} = \sum_i m_i \mathbf{a}_i = \sum_i \mathbf{F}_i = \mathbf{F}, \tag{4}$$

which introduced quantities related to the center of mass/inertia/gravity of the system. Similarly, the total angular momentum $\mathbf{L}$ of the system with respect to the origin can be written,

$$\mathbf{L} = \sum_i \mathbf{L}_i = \sum_i m_i\,\mathbf{x}_i \times \mathbf{v}_i$$

constant in the absence of external torque, but he did not develop it further.

For example, while Newton discussed **Kepler**'s 2nd law (p. 668 of [24]) that the orbits of planets around the Sun sweep out equal areas in equal times, he did not relate this to conservation of angular momentum.

[4] See, for example, [27].

[5] Euler's works are available at http://eulerarchive.maa.org/.

[6] The notation of $\mathbf{p}$ for (linear) momentum and $\mathbf{L}$ for angular momentum is common in contemporary physics literature, but an older convention is that $\mathbf{L}$ and $\mathbf{H}$ represent linear and angular momentum, respectively (perhaps $\mathbf{H}$ is for Hayward, as on p. 8 of [21] (1856)). The latter convention is still used to some extent in the engineering literature. Occasionally, $\mathbf{L}$ or $\boldsymbol{\Gamma}$ (upside-down $\mathbf{L}$) or $\mathbf{N}$ represents torque.

$$= \sum_i m_i\,(\mathbf{x}_i - \mathbf{x}_{\rm cm}) \times \mathbf{v}_i + \sum_i m_i\,\mathbf{x}_i \times \mathbf{v}_{\rm cm} = \mathbf{L}_{\rm cm} + \mathbf{x}_{\rm cm} \times m\,\mathbf{v}_{\rm cm}, \tag{5}$$

$$= \sum_i m_i\,(\mathbf{x}_i - \mathbf{x}_{\rm cm}) \times (\mathbf{v}_i - \mathbf{v}_{\rm cm}) + \sum_i m_i\,\mathbf{x}_i \times \mathbf{v}_{\rm cm} + \mathbf{x}_{\rm cm} \times \sum_i m_i\,\mathbf{v}_i - \sum_i m_i\,\mathbf{x}_{\rm cm} \times \mathbf{v}_{\rm cm}$$

$$= \mathbf{L}'_{\rm cm} + m\,\mathbf{x}_{\rm cm} \times \mathbf{v}_{\rm cm} + \mathbf{x}_{\rm cm} \times m\,\mathbf{v}_{\rm cm} - m\,\mathbf{x}_{\rm cm} \times \mathbf{v}_{\rm cm} = \mathbf{L}'_{\rm cm} + \mathbf{x}_{\rm cm} \times m\,\mathbf{v}_{\rm cm}, \tag{6}$$

where the angular momenta $\mathbf{L}_{\rm cm}$ and $\mathbf{L}'_{\rm cm}$ with respect to the center of mass are defined by,

$$\mathbf{L}_{\rm cm} = \sum_- m_i\,(\mathbf{x}_i - \mathbf{x}_{\rm cm}) \times \mathbf{v}_i, \qquad \mathbf{L}'_{\rm cm} = \sum_- m_i\,(\mathbf{x}_i - \mathbf{x}_{\rm cm}) \times (\mathbf{v}_i - \mathbf{v}_{\rm cm}), \qquad \mathbf{L}_{\rm cm} = \mathbf{L}'_{\rm cm}. \tag{7}$$

That is, while the angular momentum with respect to the center of mass could be defined two different ways, using the "absolute" velocity $\mathbf{v}_i$ or the "relative" velocity $\mathbf{v}_i - \mathbf{v}_{\rm cm}$ for mass i, the value of this quantity is the same with either definition.[7]

Equation (5) (first written by Euler) is the familiar decomposition of the total angular momentum with respect to the origin as the sum of the ("spin") angular momentum $\mathbf{L}_{\rm cm}$ with respect to the center of mass plus the ("orbital") angular momentum $\mathbf{x}_{\rm cm} \times m\mathbf{v}_{\rm cm}$ of the system (with respect to the origin) as if the mass were concentrated at the center of mass.

The time dependence of the total angular momentum $\mathbf{L}$ is related by,

$$\frac{d\mathbf{L}}{dt} = \sum_i \frac{d\mathbf{L}_i}{dt} = \sum_i \mathbf{x}_i \times \mathbf{F}_i = \sum_i \boldsymbol{\tau}_i = \boldsymbol{\tau}, \tag{8}$$

where the total torque $\boldsymbol{\tau}$ is defined with respect to the origin (which is fixed in an inertial frame).[8]

Euler also noted that eqs. (4) and (5) imply that,

$$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{L}_{\rm cm}}{dt} + \mathbf{x}_{\rm cm} \times m\,\mathbf{a}_{\rm cm} = \boldsymbol{\tau}, \tag{9}$$

$$\frac{d\mathbf{L}_{\rm cm}}{dt} = \boldsymbol{\tau} - \mathbf{x}_{\rm cm} \times m\,\mathbf{a}_{\rm cm} = \sum_i \mathbf{x}_i \times \mathbf{F}_i - \mathbf{x}_{\rm cm} \times \sum_i \mathbf{F}_i = \sum_i (\mathbf{x}_i - \mathbf{x}_{\rm cm}) \times \mathbf{F}_i = \boldsymbol{\tau}_{\rm cm}. \tag{10}$$

That is, that rate of change of angular momentum with respect to the center of mass equals the torque with respect to that point, even if it is moving.[9]

Thus, Euler developed two torque analyses, with respect to the fixed origin, and with respect to the center of mass, which are adequate for a large class of examples, and are the only torque analyses recommended (perhaps wisely) in many textbooks. Nonetheless, people have considered torque analyses with respect to a general point P, which may or may not in motion (with respect to the inertial "laboratory" frame). Such analyses involve subtleties that have led to many books and papers [32]-[75] purporting to provide needed clarifications, but which appear to have fallen short of a crisp resolution of the difficulties.[10]

[7] When considering angular momentum with respect to other moving points than the center of mass, the two definitions lead to different quantities, as discussed in sec. 3 below.

[8] Equations (4) and (8) appear on p. 224 of [31], with $\mathbf{F} + (P, Q, R)$ and $\boldsymbol{\tau} = -(S, T, U)$.

[9] Equation (10) appears on p. 228 of [31], with $d\mathbf{L}_{\rm cm}/dt$ expressed in terms of the inertia tensor.

[10] The present note may well be a member of this class.

3 Torque Analysis about a General Point P

In addition to considering angular momentum torque with respect to the origin and to the center of mass, one may wish to consider them relative to a general point P that may or may not be in motion in the inertial lab frame.

The torque $\boldsymbol{\tau}_P$ about point P is related to the torque $\boldsymbol{\tau}$ about the origin by,

$$\boldsymbol{\tau}_P = \sum_i (\mathbf{x}_i - \mathbf{x}_P) \times \mathbf{F}_i = \boldsymbol{\tau} - \mathbf{x}_P \times \mathbf{F} = \boldsymbol{\tau} - \mathbf{x}_P \times m\,\mathbf{a}_{\rm cm}. \tag{11}$$

Unfortunately, the angular momentum with respect to a point P can be defined in two ways, called absolute and relative in [66].[11]

3.1 First Definition: The Absolute Angular Momentum $\mathbf{L}_P$

We define $\mathbf{L}_P$ to be the angular momentum with respect to P, ignoring possible motion of P,

$$\mathbf{L}_P = \sum_i (\mathbf{x}_i - \mathbf{x}_P) \times m_i\,\mathbf{v}_i = \mathbf{L} - \mathbf{x}_P \times m\,\mathbf{v}_{\rm cm} = \mathbf{L}_{\rm cm} + (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\,\mathbf{v}_{\rm cm}. \tag{12}$$

However, when considering $d\mathbf{L}_P/dt$, one should take the possible velocity $\mathbf{v}_P$ into account,

$$\begin{aligned} \frac{d\mathbf{L}_P}{dt} &= \frac{d\mathbf{L}}{dt} - \mathbf{x}_P \times m\,\mathbf{a}_{\rm cm} - \mathbf{v}_P \times m\,\mathbf{v}_{\rm cm} = \boldsymbol{\tau}_P + m\,\mathbf{v}_{\rm cm} \times \mathbf{v}_P \\ &= \frac{d\mathbf{L}_{\rm cm}}{dt} - \mathbf{v}_P \times m\,\mathbf{v}_{\rm cm} + (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\,\mathbf{a}_{\rm cm}. \end{aligned} \tag{13}$$

This equation of motion is counterintuitive in that the term $m\mathbf{v}_{\rm cm} \times \mathbf{v}_P$ is neither a torque nor a rate of change of angular momentum.

3.1.1 Conditions that $d\mathbf{L}_P/dt = \boldsymbol{\tau}_P$

(a) $\mathbf{v}_P = 0$, or,

(b) $\mathbf{v}_{\rm cm} = 0$, or,

(c) $\mathbf{v}_P \parallel \mathbf{v}_{\rm cm}$.

If the system is a rigid body in contact with a fixed surface at point P, then condition (c) is satisfied when its center of mass is at the (instantaneous) center of curvature of the rolling motion, *i.e.*, if the center of mass of the rigid body is on a symmetry axis. Since this condition is satisfied in many simple examples of rigid-body motion, it is easy to form the mistaken impression that $d\mathbf{L}_P/dt = \boldsymbol{\tau}_P$ always. One way to avoid this error is to use $\mathbf{L}_P$ only in examples where conditions (a) and/or (b) are satisfied.

[11] Absolute and relative angular momentum with respect to a moving point may have been first defined (without these names) by **Wilson** in eqs. (8) and (9) of [41] (1915).

3.2 Second Definition: The Relative Angular Momentum $\mathbf{L}'_P$

If point P is moving in the lab frame, one can also define the angular momentum with respect to P similarly to eq. (7) for that with respect to the center of mass,[12]

$$\begin{aligned}\mathbf{L}'_P &= \sum_i (\mathbf{x}_i - \mathbf{x}_P) \times m_i (\mathbf{v}_i - \mathbf{v}_P) = \mathbf{L} - \mathbf{x}_{\rm cm} \times m\,\mathbf{v}_P - \mathbf{x}_P \times m\,(\mathbf{v}_{\rm cm} - \mathbf{v}_P) \\ &= \mathbf{L}_{\rm cm} + (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\,(\mathbf{v}_{\rm cm} - \mathbf{v}_P) = \mathbf{L}_P - (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\,\mathbf{v}_P. \qquad (14)\end{aligned}$$

The time derivative of $\mathbf{L}'_P$ is, recalling eq. (13),

$$\begin{aligned}\frac{d\mathbf{L}'_P}{dt} &= \frac{d\mathbf{L}_{\rm cm}}{dt} + (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\,(\mathbf{a}_{\rm cm} - \mathbf{a}_P) \qquad (15) \\ &= \frac{d\mathbf{L}_P}{dt} - \mathbf{v}_{\rm cm} \times m\,\mathbf{v}_P - (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\,\mathbf{a}_P = \boldsymbol{\tau}_P + (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times (-m\,\mathbf{a}_P) \equiv \boldsymbol{\tau}'_P.\end{aligned}$$

Since the moving point P defines a (nonrotating) accelerated frame, the equation of motion of the angular momentum $\mathbf{L}'_P$ relative to this moving point includes the "fictitious" torque associated with the "fictitious" (coordinate) force $-m\,\mathbf{a}_P$ that appears in this frame to act on the center of mass of the system (as anticipated by **d'Alembert** (1743) [76]).[13]

3.2.1 Conditions that $d\mathbf{L}'_P/dt = \boldsymbol{\tau}_P$

(d) $(\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times \mathbf{a}_P = 0$, for which special cases are,

(e) $\mathbf{v}_P$ is independent of time (as for steady rolling), or

(f) $\mathbf{x}_P = \mathbf{x}_{\rm cm}$ (as holds for rotation of a rigid body about a fixed, symmetry axis).

3.2.2 Conditions that $\mathbf{L}'_P = \mathbf{L}_P$

(g) $(\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times \mathbf{v}_P = 0$, for which special cases are,

(a) $\mathbf{v}_P = 0$, or,

(f) $\mathbf{x}_P = \mathbf{x}_{\rm cm}$.

[12] Even when point P is moving, one can consider use of angular momentum $\mathbf{L}_P$ rather than $\mathbf{L}'_P$, so if the angular momentum with respect to point P is not clearly defined, confusion can result.

[13] Nov. 16, 2024. In principle, we could suppose that the reference point P of the torque analysis is associated with a rotating coordinate system with an arbitrary angular velocity $\boldsymbol{\Omega}(t)$ with respect to the lab frame. Then, we should consider the "fictitious" forces $m\mathbf{r} \times \dot{\boldsymbol{\Omega}} + 2m\mathbf{v} \times \boldsymbol{\Omega} + m\boldsymbol{\Omega} \times (\mathbf{r} \times \boldsymbol{\Omega})$ that would act on the center of mass of the system (at $\mathbf{r}$, with velocity $\mathbf{v}$, in the rotating frame). See, for example, eq. (39.7) of [77] or pp. 168-172 of [78]. This suggests that we should also consider the "fictitious" torques associated with the "fictitious" forces related to the arbitrary angular velocity $\boldsymbol{\Omega}$. However, doing so would, in general, change the result of the torque analysis (as expressed in lab-frame variables), which must be independent of the choice of $\boldsymbol{\Omega}$. Hence, it must be that the supposed "fictitious" torques do not exist (although the "fictitious" forces do exist). *General relativity is complicated!* Thus, in a torque analysis it suffices to suppose the axes are not rotating. See also [79].

3.2.3 Conditions that $d\mathbf{L}'_P/dt = d\mathbf{L}_P/dt$

(h) $\mathbf{v}_{\rm cm} \times m\,\mathbf{v}_P + (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\,\mathbf{a}_P = 0$, for which special cases are,

(f) $\mathbf{x}_P = \mathbf{x}_{\rm cm}$, or,

(i) $\mathbf{v}_P = 0$ and $\mathbf{x}_{\rm cm} - \mathbf{x}_P \perp \mathbf{a}_P$, as holds for a rigid body, with both a symmetry plane and a symmetry axis, that also rolls without slipping on a fixed surface.

These conditions reinforce the "conventional wisdom" that torque analysis are best made about a fixed point P or about the center of mass.

In particular, it is disconcerting that for a rigid body which rolls without slipping on a fixed surface, the point P of contact is instantaneously at rest if it is regarded as being fixed with respect to the body, and $\mathbf{v}_P = 0$, so $\mathbf{L}'_P = \mathbf{L}_P$, but it can be that $d\mathbf{L}'_P/dt \neq d\mathbf{L}_P/dt$ (see, for example, secs. A.5.3 and A.5.5 below). Hence, **it is not prudent to calculate $\mathbf{L}'_P$ or $\mathbf{L}_P$ first, and then take their time derivatives for use in a torque analysis. It seems better to compute the time derivatives directly from eq. (13) or (15).**

3.3 Spin and Orbital Kinetic Energy and Angular Momentum

We have seen in eqs. (5)-(6) that the total angular momentum $\mathbf{L}$ with respect to the origin can be decomposed as the sum of the "spin" angular momentum $\mathbf{L}_{\rm cm} = \mathbf{L}'_{\rm cm}$ with respect to the center of mass plus the "orbital" angular momentum $\mathbf{x}_{\rm cm} \times m\mathbf{v}_{\rm cm}$ of the system (with respect to the origin) as if the mass were concentrated at the center of mass.

A similar decomposition holds for the total kinetic energy T of the system,

$$\begin{aligned} T &= \frac{1}{2}\sum_i m_i \mathbf{v}_i^2 = \frac{1}{2}\sum_i m_i[\mathbf{v}_{\rm cm} + (\mathbf{v}_i - \mathbf{v}_{\rm cm})]^2 \\ &= \frac{1}{2}\sum_i m_i \mathbf{v}_{\rm cm}^2 + \sum_i m_i \mathbf{v}_{\rm cm}\cdot(\mathbf{v}_i - \mathbf{v}_{\rm cm}) + \frac{1}{2}\sum_i m_i(\mathbf{v}_i - \mathbf{v}_{\rm cm})^2 \\ &= \frac{1}{2} m\mathbf{v}_{\rm cm}^2 + \frac{1}{2}\sum_i m_i(\mathbf{v}_i - \mathbf{v}_{\rm cm})^2, \end{aligned} \tag{16}$$

which is the sum of the kinetic energy of the "orbital" motion of the center of mass as if all mass were concentrated there, and the kinetic energy of the "spin" motion relative to the center of mass.

The question as to whether there are other points P for which similar decompositions occur was addressed by **Wilson** (1915) [41]. His eqs. (8) and (9) are the equivalent of our eqs. (12) and (14), which indicate the only when point P is at the center of mass does the desired decomposition of angular momentum occur.[14]

He also noted that if $\mathbf{v}_{\rm cm}$ were replaced by $\mathbf{v}_P$ in our eq. (16), the desired decomposition of kinetic energy would hold provided $\mathbf{v}_P \cdot (\sum_i m_i\mathbf{v}_i - m\mathbf{v}_P) = m\mathbf{v}_P \cdot (\mathbf{v}_{\rm cm} - \mathbf{v}_P) = 0$, which is satisfied by all points on a sphere of diameter $|\mathbf{x}_{\rm cm}|$ that has one "pole" at the origin and

[14] Wilson may have been the first give expressions for the absolute and relative angular momentum with respect to a general point P. Wilson also gave the equivalent of our torque equation (15) for relative angular momentum in his eq. (12), but he did not give the equivalent of our eq. (13) for absolute angular momentum.

the other at the center of mass. However, the only point of interest on this sphere beside the origin is the center of mass (such that Wilson's "theorem" is little known).

4 Rigid Bodies

Thus far, the system of point masses m_i has been rather general, *i.e.*, gases, liquids and solids. In classical mechanics, one typically specializes to the case of solids that can be approximated as rigid, meaning that the (scalar) distance between any two constituent masses is independent of time.

This approximation implies an infinite speed of sound, and hence there are no rigid bodies in Nature.

An additional attribute of classical rigid bodies is that their internal forces sum to zero,

$$\mathbf{F} = \sum \mathbf{F}_i = \sum_i \mathbf{F}_{i,\text{ext}} + \sum_{i\neq j} \mathbf{F}_{i,j} = \sum_i \mathbf{F}_{i,\text{ext}} = \mathbf{F}_{\text{ext}}, \tag{17}$$

such that the total force on the system is that due only to external forces. This follows from Newton's 3rd law (p. 83 of [1]): *To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts*. Indeed, eq. (17) follows from a weaker version of the 3rd law, which does not requires that the *mutual action of two bodies upon one another* be along their line of centers, but only that $\mathbf{F}_{i,j} = -\mathbf{F}_{j,i}$.[15]

If one accepts the stronger form of Newton's 3rd law, then it also follows that the torque with respect to the fixed origin is,[16]

$$\begin{aligned} \boldsymbol{\tau} &= \sum \mathbf{x}_i \times \mathbf{F}_i = \sum_i \mathbf{x}_i \times \mathbf{F}_{i,\text{ext}} + \frac{1}{2}\sum_{i\neq j} [\mathbf{x}_i \times \mathbf{F}_{i,j} + \mathbf{x}_j \times \mathbf{F}_{j,i}] \\ &= \sum_i \mathbf{x}_i \times \mathbf{F}_{i,\text{ext}} + \frac{1}{2}\sum_{i\neq j} (\mathbf{x}_i - \mathbf{x}_j) \times \mathbf{F}_{i,j} = \sum_i \mathbf{x}_i \times \mathbf{F}_{i,\text{ext}} = \boldsymbol{\tau}_{\text{ext}}. \end{aligned} \tag{18}$$

Then, the equations of motion for a rigid body can be written as,

$$m\,\mathbf{a}_{\text{cm}} = \mathbf{F}_{\text{ext}}, \qquad \frac{d\mathbf{L}}{dt} = \boldsymbol{\tau}_{\text{ext}}, \tag{19}$$

where m is the mass of the rigid body and $\mathbf{L}$ is its angular momentum with respect to the origin.[17]

[15] This weaker form is the typical interpretation of Newton's 3rd law.

[16] The result (18) may have been first explicitly stated by **Poisson**, pp. 447-448 of [80] (1833).

[17] The charged particles in a rigid body are presumably held together by electromagnetic forces, but the magnetic part of the **Lorentz** force law does not obey Newton's 3rd law, with the violation being of order v^2/c^2, where v is the velocity of the charges and c is the speed of light in vacuum. That is, for electric charges q_i at positions $\mathbf{x}_i$ with velocities $\mathbf{v}_i$ with $v_i \ll c$, the Lorentz force is, in Gaussian units,

$$\mathbf{F}_{1,2} = q_1\mathbf{E_2} + q_1\frac{\mathbf{v}_1}{c} \times \mathbf{B}_2 = q_1 q_2 \frac{\mathbf{x}_{12}}{x_{12}^3} + q_1\frac{\mathbf{v}_1}{c} \times \left(q_2\frac{\mathbf{v}_2}{c} \times \frac{\mathbf{x}_{12}}{x_{12}^3}\right)$$

It is agreeable that there are six scalar equations for motion for a rigid body, which has six degrees of freedom. The latter insight is contained in the arguments of **Mozzi** [81] and **Chasles** [82, 83] that any motion of a rigid body can be regarded as the combination of a translation of a point P fixed in the body and a rotation about that point. As a consequence, the velocity $\mathbf{v}$ of a mass element dm centered on point $\mathbf{x}$ in the body can be written as,

$$\mathbf{v} = \mathbf{v}_P + \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{x}_P), \tag{22}$$

where the angular velocity $\boldsymbol{\omega}$ (with respect to the inertial lab frame) is independent of point P.

4.1 The Inertia Tensor

As noted by Euler in chap. 7, p. 166, of [30], the relative angular momentum $\mathbf{L}'_Q$ of a rigid body with respect to a point P in that body can be written as,

$$\begin{aligned} \mathbf{L}'_P &= \int (\mathbf{x} - \mathbf{x}_P) \times dm\,(\mathbf{v} - \mathbf{v}_P) = \sum_i m_i\,(\mathbf{x} - \mathbf{x}_P) \times [\boldsymbol{\omega} \times (\mathbf{x} - \mathbf{x}_P)] \\ &= \int dm\,\{(\mathbf{x} - \mathbf{x}_P)^2 \boldsymbol{\omega} - [(\mathbf{x} - \mathbf{x}_P) \cdot \boldsymbol{\omega}](\mathbf{x} - \mathbf{x}_P)\} \equiv \mathsf{I}_P \cdot \boldsymbol{\omega}, \end{aligned} \tag{23}$$

where the inertia tensor I_P has components $I_{P,ij}$ given by,

$$I_{P,ij} = \int dm\,[\delta_{ij}(\mathbf{x} - \mathbf{x}_P)^2 - (x_i - x_{P,i})(x_j - x_{P,j})]. \tag{24}$$

Euler emphasized the case that point P is at the center of mass of the rigid body, for which its kinetic energy T can be written as, noting that $\int dm\,(\mathbf{x} - \mathbf{x}_{\rm cm}) = 0$,

$$\begin{aligned} T &= \frac{1}{2}\int dm\,v^2 = \frac{1}{2}\int dm\,[\mathbf{v}_{\rm cm} + \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{x}_{\rm cm})]^2 \\ &= \frac{m\,v_{\rm cm}^2}{2} + \int dm\,\mathbf{v}_{\rm cm} \cdot [\boldsymbol{\omega} \times (\mathbf{x} - \mathbf{x}_{\rm cm})] + \frac{1}{2}\int dm\,[\boldsymbol{\omega} \times (\mathbf{x} - \mathbf{x}_{\rm cm})]^2 \\ &= \frac{m\,v_{\rm cm}^2}{2} + \frac{1}{2}\int dm\,\{[\omega^2(\mathbf{x} - \mathbf{x}_{\rm cm})^2 - [\boldsymbol{\omega} \cdot (\mathbf{x} - \mathbf{x}_{\rm cm})]^2\} = \frac{m\,v_{\rm cm}^2}{2} + \frac{\boldsymbol{\omega} \cdot \mathsf{I}_{\rm cm} \cdot \boldsymbol{\omega}}{2}, \end{aligned} \tag{25}$$

$$= q_1 q_2 \left[\frac{\mathbf{x}_{12}}{x_{12}^3}\left(1 - \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}\right) + \frac{\mathbf{v}_1 \cdot \mathbf{x}_{12}}{x_{12}^3}\mathbf{v}_2\right], \tag{20}$$

$$\begin{aligned} \mathbf{F}_{2,1} &= q_1 q_2 \left[\frac{\mathbf{x}_{21}}{x_{21}^3}\left(1 - \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}\right) + \frac{\mathbf{v}_2 \cdot \mathbf{x}_{21}}{x_{21}^3}\mathbf{v}_1\right] = -q_1 q_2 \left[\frac{\mathbf{x}_{12}}{x_{12}^3}\left(1 - \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}\right) + \frac{\mathbf{v}_2 \cdot \mathbf{x}_{12}}{x_{12}^3}\mathbf{v}_1\right] \\ &= -\mathbf{F}_{1,2} + q_1 q_2 \frac{(\mathbf{v}_2 \cdot \mathbf{x}_{12})\mathbf{v}_1 - (\mathbf{v}_1 \cdot \mathbf{x}_{12})\mathbf{v}_2}{x_{12}^3}, \end{aligned} \tag{21}$$

where $\mathbf{x}_{12} = \mathbf{x}_1 - \mathbf{x}_2 = -\mathbf{x}_{21}$. Thus, when $\mathbf{v}_1$ is different from $\mathbf{v}_2$ and neither is along the line of centers $\mathbf{x}_{12}$, we have that $\mathbf{F}_{12} \neq -\mathbf{F}_{12}$.

This was recognized by **Ampère** (1822), who rejected the notion that isolated moving charges could exist, and argued (pp. 21-24 of [84]) that all moving electric charges are part of closed electric-current loops, for which he devised a magnetic force law that does obey Newton's 3$^{\rm rd}$ law. This led **Maxwell** to dub Ampère the *Newton of electricity* (Art. 528 of [85]). Some 60 years elapsed from the time of Ampère to the acceptance of the Lorentz force law, and that rigid bodies are not compatible with electromagnetism (nor with special relativity).

where $\mathsf{I}_{\rm cm}$ is the inertia tensor with respect to the center of mass of the body. Equation (25) is another form of the "spin-orbit" decomposition of the kinetic energy (sec. 3.3).

For completeness, we note that the tensor version of the "parallel-axis" theorem is,

$$I_{P,ij} = I_{{\rm cm},ij} + m\,[\delta_{ij}(\mathbf{x}_P - \mathbf{x}_{\rm cm})^2 - (x_{P,i} - x_{{\rm cm},i})(x_{P,j} - x_{{\rm cm},j})]. \tag{26}$$

For any rigid body there exist so-called principal axes with respect to which the inertial tensor is diagonal. If the body has a symmetry plane, then one principal axis is perpendicular to that plane, and if the body has a symmetry axis, then that is a principal axis.[18]

4.2 Euler's Torque Equation for a Point P Fixed in the Rigid Body

Euler also noted that, in a frame which rotates with angular velocity $\boldsymbol{\omega}$ with respect to an inertial frame, the time derivative $\delta\mathbf{A}/\delta t$ of a vector $\mathbf{A}$ is related to its time derivative $d\mathbf{A}/dt$ in the inertial frame by,

$$\frac{d\mathbf{A}}{dt} = \frac{\delta\mathbf{A}}{\delta t} + \boldsymbol{\omega}\times\mathbf{A}. \tag{27}$$

Using this, the torque equation (15) with respect to a point P fixed with respect to the rigid body becomes, recalling eq. (23),[19]

$$\boldsymbol{\tau}'_P = \frac{d\mathbf{L}'_P}{dt} = \frac{\delta\mathbf{L}'_P}{\delta t} + \boldsymbol{\omega}\times\mathbf{L}'_P = \frac{\delta\mathsf{I}_P}{\delta t}\cdot\boldsymbol{\omega} + \mathsf{I}_P\cdot\frac{\delta\boldsymbol{\omega}}{\delta t} + \boldsymbol{\omega}\times\mathsf{I}_P\cdot\boldsymbol{\omega} = \mathsf{I}_P\cdot\frac{d\boldsymbol{\omega}}{dt} + \boldsymbol{\omega}\times\mathsf{I}_P\cdot\boldsymbol{\omega}, \tag{28}$$

since $\delta\mathsf{I}_P/\delta t = 0$.

Note that if the angular velocity $\boldsymbol{\omega}$ is parallel to the principal axis defined by a symmetry axis or plane of the rigid body, then $\mathsf{I}_P\cdot\boldsymbol{\omega}$ is parallel to $\boldsymbol{\omega}$, and $\boldsymbol{\omega}\times\mathsf{I}_P\cdot\boldsymbol{\omega}$ is zero.

4.2.1 Conditions that $d\mathbf{L}'_P/dt = \mathsf{I}_P\cdot d\boldsymbol{\omega}/dt = \boldsymbol{\tau}_P$

Recalling sec. 3.2.1,

(d) $(\mathbf{x}_{\rm cm} - \mathbf{x}_P)\times\mathbf{a}_P = 0$, for which special cases are,

(e) $\mathbf{v}_P$ is independent of time (as for steady rolling), or

(f) $\mathbf{x}_P = \mathbf{x}_{\rm cm}$ (as holds for rotation of a rigid body about a fixed, symmetry axis).

and,

(j) $\boldsymbol{\omega} \parallel$ to a symmetry axis, or,

(k) $\boldsymbol{\omega} \perp$ to a symmetry plane.

[18] See, for example, sec. 32 of [77].

[19] Euler's famous equation (28) was first deduced in prob. 88, p. 342, of [30], for P at the center of mass. In that case $\mathbf{L}_P = \mathbf{L}'_P = \mathbf{L}_{\rm cm}$, $\boldsymbol{\tau}_P = \boldsymbol{\tau}'_P = \boldsymbol{\tau}_{\rm cm}$, and we could also have proceeded from eq. (13).

4.3 Rolling and/or Slipping of a Rigid Body on a Fixed Surface

We now turn to a narrower class of examples, somewhat anticipated above, in which a rigid body rolls and, in general, also slips on a fixed surface. Here, there is a point (or line) of contact between the rigid body and the surface, so it is tempting to consider analyses in which this point/line plays a role.

There is a contact force at this point/line, that is not immediately known, but which should be elucidated in a full analysis. Experience with static examples of a rigid body in contact with a surface has shown that it is often expedient to compute torques about the point/line of contact, to avoid use of the contact force in (the early parts of) the analysis. Hence, it seems natural in dynamic examples with a contact point to consider torque analyses with respect to this (generally moving) point as alternatives to the "tried and true" torque analyses with respect to a fixed origin or to the (moving) center of mass.

4.3.1 Analysis Regarding the Contact Point as Fixed in the Lab Frame

One possibility for an analysis at a given time is to regard the contact point P at that time as fixed in the lab frame.[20]

With respect to a fixed point P, the absolute angular momentum $\mathbf{L}_P$ of eq. (12) equals the relative angular momentum $\mathbf{L}'_P$ of eq. (14). So, we can take the time derivative of eq. (12) with $\mathbf{v}_P = 0$, and use eq. (11) to find,

$$\frac{d\mathbf{L}_P}{dt} = \frac{d\mathbf{L}}{dt} - \mathbf{x}_P \times m\,\mathbf{a}_{\rm cm} = \boldsymbol{\tau} - \mathbf{x}_P \times m\,\mathbf{a}_{\rm cm} = \boldsymbol{\tau}_P \tag{29}$$

$$= \frac{d\mathbf{L}_{\rm cm}}{dt} + (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\,\mathbf{a}_{\rm cm}, \tag{30}$$

$$\frac{d\mathbf{L}_{\rm cm}}{dt} = \boldsymbol{\tau}_P + (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times (-m\,\mathbf{a}_{\rm cm}) = \boldsymbol{\tau}_{\rm cm}. \tag{31}$$

Thus, a torque analysis regarding the contact point P as fixed reverts to the general form (10) for an analysis with respect to the moving center of mass (which may then benefit from Euler's equation (28)).

4.3.2 Analyses Regarding the Contact Point as Moving

Another view is to consider the contact point at a given time be fixed with respect to the body, and hence moving with time. Then, one can use either/both of the torque analyses of eqs. (13) or (15) for $\mathbf{L}_P$ and $\mathbf{L}'_P$, respectively.[21] Of these, the analysis based on $\mathbf{L}'_P$ is closer to the spirit of Euler, so if the goal is to make a nonstandard analysis, the use of $\mathbf{L}_P$ is favored.

Absolute Angular Momentum $\mathbf{L}_P$ and the Inertia Tensor

The absolute angular momentum $\mathbf{L}_P$ of eq. (12) with respect to a point P in a rigid body can be related to the inertia tensor by recalling eqs. (14) and (23),

$$\mathbf{L}_P = \mathbf{L}'_P + (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\,\mathbf{v}_P = \mathsf{I}_P \cdot \boldsymbol{\omega} + (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\,\mathbf{v}_P. \tag{32}$$

[20] This analysis has been emphasized in [74].

[21] This approach is emphasized in [73, 75], but only for the case of rolling without slipping.

This relation is not simple unless point P is at the center of mass, or the velocity $\mathbf{v}_P$ of this point is zero. The latter holds (instantaneously) when P is the point of contact of the rigid body with a fixed surface on which the body rolls without slipping (but one must note that $d\mathbf{L}_P/dt$ is not $\mathsf{I}_P \cdot d\boldsymbol{\omega}/dt$). It also holds for a spinning top with one point fixed.

This suggests that torque analyses involving the absolute angular momentum $\mathbf{L}_P$ will be of little practical use except in these special (but interesting) cases.

4.3.3 Analysis with Respect to the Instantaneous Center of Rotation

The instantaneous center of rotation of a (rotating) rigid body is that point C of the body which is instantaneously at rest, *i.e.*, $\mathbf{v}_C = 0$. As noted in secs. 3.1.1 and 3.2.1-3, for this point we have that $\mathbf{L}_C = \mathbf{L}'_C$ although not necessarily that $d\mathbf{L}_C/dt = d\mathbf{L}'_C/dt$. Nonetheless, a torque analysis with respect to the instantaneous center of rotation has some appeal.

In particular, for "planar" motion of a rigid, cylindrical body, where all velocities lie in the symmetry plane perpendicular to lines of the cylinder, the angular velocity $\boldsymbol{\omega}$ is perpendicular to this plane. We can obtain a useful torque equation for this case, following **Loney**, Art. 214, p. 287, of [35]. **The argument is delicate, in that one might suppose we could use eq. (32) with $\mathbf{v}_C = 0$ and write $\mathbf{L}_C = I_C\,\boldsymbol{\omega}$ (correct), and infer that $d\mathbf{L}_C/dt = I_C\,\dot{\boldsymbol{\omega}} + \dot{I}_C\,\boldsymbol{\omega}$ (incorrect!).** Rather, we should follow the advice at the end of sec. 3.2 above.

We recall a form of eq. (12) for the absolute angular momentum with respect to the instantaneous center C of rotation,

$$\mathbf{L}_C = \mathbf{L}_{\rm cm} + (\mathbf{x}_{\rm cm} - \mathbf{x}_C) \times m\,\mathbf{v}_{\rm cm} = I_{\rm cm}\,\boldsymbol{\omega} + (\mathbf{x}_{\rm cm} - \mathbf{x}_C) \times m\,\mathbf{v}_{\rm cm}, \tag{33}$$

$$\frac{d\mathbf{L}_C}{dt} = I_{\rm cm}\,\dot{\boldsymbol{\omega}} + (\mathbf{x}_{\rm cm} - \mathbf{x}_C) \times m\,\dot{\mathbf{v}}_{\rm cm} = \boldsymbol{\tau}_C, \tag{34}$$

as the moment of inertia $I_{\rm cm}$ of a rigid body about its center of mass is constant in time. And, recalling eq. (22), we can write the velocity of the center of mass relative to the instantaneous center C as,

$$\mathbf{v}_{\rm cm} = \boldsymbol{\omega} \times (\mathbf{x}_{\rm cm} - \mathbf{x}_C), \qquad \boldsymbol{\omega} \times \mathbf{v}_{\rm cm} = \boldsymbol{\omega} \times [\boldsymbol{\omega} \times (\mathbf{x}_{\rm cm} - \mathbf{x}_C)] = -\omega^2(\mathbf{x}_{\rm cm} - \mathbf{x}_C), \tag{35}$$

since $\boldsymbol{\omega} \cdot (\mathbf{x}_{\rm cm} - \mathbf{x}_C) = 0$ for planar motion, if we take $\mathbf{x}_C$ to be in the symmetry plane. Then, eq. (34) can be written as, noting that $\boldsymbol{\omega} \cdot \mathbf{v}_{\rm cm} = 0$ and $\mathbf{v}_{\rm cm}^2 = (\mathbf{x}_{\rm cm} - \mathbf{x}_C)^2\,\omega^2$,

$$\begin{aligned}\frac{d\mathbf{L}_C}{dt} &= I_{\rm cm}\,\dot{\boldsymbol{\omega}} + m\,\dot{\mathbf{v}}_{\rm cm} \times \frac{\boldsymbol{\omega} \times \mathbf{v}_{\rm cm}}{\omega^2} = I_{\rm cm}\,\dot{\boldsymbol{\omega}} + \frac{m\,(\mathbf{v}_{\rm cm} \cdot \dot{\mathbf{v}}_{\rm cm})}{\omega}\,\hat{\boldsymbol{\omega}} = \frac{1}{2\omega}\frac{d}{dt}\left[I_{\rm cm}\omega^2 + m\mathbf{v}_{\rm cm}^2\right]\hat{\boldsymbol{\omega}} \\ &= \frac{1}{2\omega}\frac{d}{dt}\left[I_{\rm cm}\,\omega^2 + m(\mathbf{x}_{\rm cm} - \mathbf{x}_C)^2\,\omega^2\right]\hat{\boldsymbol{\omega}} = \frac{1}{2\omega}\frac{d}{dt}\left(I_C\,\omega^2\right)\hat{\boldsymbol{\omega}} = I_C\,\dot{\boldsymbol{\omega}} + \frac{1}{2}\dot{I}_C\,\boldsymbol{\omega} = \boldsymbol{\tau}_C,\end{aligned} \tag{36}$$

using that $I_C = I_{\rm cm} + m(\mathbf{x}_{\rm cm} - \mathbf{x}_C)^2$ according to the parallel-axis theorem.

If the rigid, cylindrical body has a symmetry axis as well as the symmetry plane mentioned above, then the moment of inertia is the same about any possible point of contact on the cylindrical surface of the body, such that $\dot{I}_C = 0$ and,

$$I_C\,\dot{\boldsymbol{\omega}} = \boldsymbol{\tau}_C \qquad \text{(symmetry plane and symmetry axis).} \tag{37}$$

Chirgwin amd Plumpton, p. 279 of [51] (and also **Tiersten** [65]) provided a shorter derivation, starting from the work-energy relation that as the rigid body rotates by small angle $d\theta$ during time dt about the instantaneous center C, the torque $\boldsymbol{\tau}_C$ about that point does work $\tau_C\, d\theta$, which changes the kinetic energy $T = I_C\,\omega^2/2$ by dT, according to,

$$\tau_C \frac{d\theta}{dt} = \tau_C\,\omega = \frac{dT}{dt} = \frac{d}{dt}\frac{I_C\,\omega^2}{2} = I_C\,\omega\dot{\omega} + \frac{\dot{I}_C\omega^2}{2}\,, \tag{38}$$

and hence the scalar form of eq (36) follows on dividing eq. (38) by ω.

Some care is required in torque analyses about the instantaneous center of rotation, as emphasized in [39] (which also discussed limits to the use of the torque equation $\boldsymbol{\tau}_P = I_P\,\dot{\boldsymbol{\omega}}$ with respect to some point P).[22]

4.3.4 Rolling without Slipping of a Rigid Body on a Fixed Surface

An oft-discussed special case is rolling without slipping of a rigid body on a fixed surface. In this case the point P of contact is also the instantaneous center of rotation, and the comments of sec. 4.3.3 apply.

5 Partial Survey of Torque Analyses since 1900

Routh ($\approx$ 1900), Art. 134 of [32], followed Euler's recommendation that torque analyses be performed with respect to a fixed point or to the center of mass. Some discussion, inspired by Poinsot, of the instantaneous center of rotation appears in Chap. IV of [33], but he did not develop a torque analysis about this center in sense of our sec. 4.3.3.

Whittaker (1904) emphasized analyses with respect to the instantaneous center of rotation, starting on pp. 2-3 of [34].

A successor to Routh was **Loney**, who discussed (not very clearly) the angular momentum $\mathbf{L}'_P$ relative to a moving point in Art. 192, p. 243 of [35] (1909), along with the advice: *The use of the expressions of this article often simplifies the solution of a problem; but the beginner is very liable to make mistakes, and, to begin with, at any rate, he would do well to confine himself to the formulae of Art. 187* (*i.e.*, to a torque analysis about the center of mass). Loney discussed a uniform sphere that rolls without slipping on an incline via a torque analysis about the center of mass in Art. 194, p. 244; a variable mass cylinder that rolls down a snowy slope in Art. 203, p. 261; and torque analysis about the instantaneous center of rotation in Art. 214, p. 287.

Besant (1914) [38] discussed the angular momentum $\mathbf{h}$ with respect to a moving point P in Art. 316, p. 349, in a manner equivalent to the absolute angular momentum $\mathbf{L}_P$ of our sec. 3.1. He also discussed a torque analysis, based on Euler's torque equation (28), about the instantaneous center of rotation of a rigid body in Art. 320, p. 352, arriving at our eq. (36).

Huntington (1914) [39] discussed subtleties in torque analyses with respect to the instantaneous center of rotation. As examples of misleading torque analyses, Huntington cited

[22] A torque analysis by the author that uses the instantaneous center of rotation is in sec. 2.2 of [88].

Smith and Longley, p. 236 of [36]; **Dadourian**, secs. 184 and 188 of [37]; and **Fuller and Johnston**, pp. 308-309 of [40].

Wilson (1915) [41] responded to Huntington's paper [39] with what may be the first derivation of the torque analyses for both absolute and relative angular momentum about a general, moving point.

Ramsey (1929) discussed a torque analysis about the instantaneous center of rotation in sec. 16.6. p. 239, Vol. 1, of [42], in the manner of Loney [35].

Den Hartog (1948) advocated torque analyses about a fixed point or about the center of mass in his Chap. 13, but briefly considered use of the instantaneous center C of rotation on pp. 244-246, using as an example a cylinder whose center of mass does not lie on its axis. He warned the reader on p. 246 (with additional remarks on p. 300) that, in general, $\tau_C \neq I_C\dot{\omega}$, but he did not provide a general prescription for use of the instantaneous center.

Milne (1948), chap. 8 of [44], presented definitions of absolute and relative angular momentum, $\mathbf{H}(O)$ and $\mathbf{H}_r(O)$, respectively, with respect to a point O, but did not develop these into the torque analyses (13) and (15).[23]

Osgood (1949) discussed the relative angular momentum $\boldsymbol{\sigma}'_r$ with respect to point O' in sec. VI.8, p. 202, of [45].[24] The torque analysis for relative angular momentum (our eq. (15) was expressed somewhat indirectly in eq. (5) of sec. VI.9, p. 204, and a bit differently in eq. (5) of sec. VI.10, p. 206. Then, sec. VI.11. pp. 207-208. presented a torque analysis for planar motion about the instantaneous center C of rotation, noting that the simple form $I_C\,\dot{\omega} = \tau_C$ holds only for rigid bodies with a symmetry axis (as well as a symmetry plane). Also, condition (d) of our sec. 4.2.1, that $(\mathbf{x}_{\rm cm} - \mathbf{x}_P) \perp \mathbf{a}_{\rm cm}$, was shown by Osgood on p. 164 to be equivalent to,

(d′) The instantaneous center of rotation is at a fixed distance from the center of mass at all times.

Becker (1954) emphasized the instantaneous center C of rotation for planar motion in sec. 9-3, p. 194, of [49]. He continued this discussion in sec. 9-11, p. 207, noting that in general one cannot write $\tau_C = I_C\,\dot{\omega}$, but that this does hold for rigid bodies with both a symmetry plane and a symmetry axis.

Lambe (1958) discussed the relative angular momentum in sec. 5.9, p. 150, of [46], and gave the torque analysis for this in eq. 1, p. 151 for rigid bodies with a symmetry plane. Then, in secs. 5.10, p. 151, and sec. 5.13, p. 155, he discussed a torque analysis with respect to the instantaneous center of rotation in the manner of Loney [35], and used this method for the example, p. 156, of a cylinder rolling without slipping down an incline (where the point of contact is the instantaneous center of rotation).

McCuskey (1959) discussed torque analysis for angular momentum $\mathbf{L}_O$ with respect to point O in a moving rigid body, sec. 4-1 of [47], without clarifying whether this involved the absolute or relative angular momentum. He then hinted that this should be done only if point O is at the center of mass, or is instantaneously at rest.

Symon considered the relative angular momentum $\mathbf{L}_Q$ with respect to a moving point Q, more clearly in sec. 4.2 of the 2nd edition (1960) of his textbook [48] than in the 1st. See his

[23] Appendix V of [90] transcribes Milne's chap. 8 into the notation used here.

[24] His notation would have been clearer if he had written $\boldsymbol{\sigma}_{O'}$ rather than $\boldsymbol{\sigma}'_r$, and $\mathbf{r}_{O'}$ and $\mathbf{v}_{O'}$ for the position and velocity of point O' rather than $\mathbf{r}_0$ and $\mathbf{v}_0$.

eqs. (4-19) and (4-23). The torque equation for $\mathbf{L}_Q$ was given in his eq. (4-25), p. 159, which includes the term $\sum_k (\mathbf{r}_k - \mathbf{r}_Q) \times \mathbf{F}_k^i$ due to the internal force $\mathbf{F}_k^i$ on particle k. This term is usually considered to vanish in classical mechanics, based on the strong form of Newton's 3rd law as discussed in sec. 4 above. Symon was a particle accelerator physicist (as is the present author), who worked with beams of energetic charge particles for which this strong form does not hold. However, after some brief discussion on p. 160, Symon reverted to the usual acceptance of the strong form of the 3rd law in textbooks on mechanics

Yeh and Abrams (1960) discussed the absolute angular momentum $\mathbf{L}_O$ about a point O in eq. (10-6a), p. 227 of [50], and the relative angular momentum $\mathbf{L}_{O'}$ about a point O' in eq. (10-6c). They added that "to avoid future confusion", it is best to restrict the concept of absolute angular momentum to that about a fixed point O (in an inertial frame). Their eq. (10-10), p. 230, corresponds to our torque equation (15) for relative angular momentum.

Chirgwin and Plumpton (1963) [51] appealed to both physicists and engineers by writing $\mathbf{p}$ for linear momentum and $\mathbf{h}$ for angular momentum. On p. 221 they defined the absolute angular momentum $\mathbf{h}(A)$ about a general point A (corresponding to $\mathbf{L}_A$ of our eq. (12)), and the relative angular momentum $\mathbf{h}_r(A)$ (corresponding to $\mathbf{L}'_A$ of our eq. (14)). They did not, however, deduce general torque equations for $\mathbf{h}(A)$ and $\mathbf{h}_r(A)$, but considered only the special cases that for a fixed point A then $d\mathbf{h}(A)/dt = \mathbf{\Gamma}(A)$ $(= \boldsymbol{\tau}_A)$, and that when A is at the center of mass G, then $d\mathbf{h}(G)/dt = d\mathbf{h}_r(G)/dt = \mathbf{\Gamma}(G)$. They discussed torque analyses about the instantaneous center of rotation in sec. 8:6, p. 278, and applied this to a half-cylindrical shell on p. 280.

Wirsching and Murdock (1965) considered the absolute angular momentum $\mathbf{H}_P$ about a point P in sec. 6-2, p. 166, of [52], and stated the equivalent of our eq. (13) in their eq. (6.58), p. 171.

Spiegel (1967, in a *Schaum's Outline*) considered the torque equation for a moving point P in prob. 7.85, p. 191, of [53], leaving the details as an "exercise for the student".

Huang (1968) discussed both the absolute angular momentum $\mathbf{H}_A$ with respect to a moving point A, as well as the relative angular momentum $(\mathbf{H}_A)_r$, in eqs. (9-24) and (9.25), p. 713, of [54], respectively. The torque equations for these two angular momenta were given in eq. (9-31), p. 717, followed by a discussion of the conditions described in our secs. 3.1.1 and 3.2.1 above.

A use of the absolute angular momentum in an introductory text is implicit on p. 249 of the 2nd edition (1973) **Berkeley Mechanics** course [55] in the statement: *we shall require the moment of force about P to equal the rate of change of angular momentum about P (P being the point of contact).*[25]

Melissinos and Lobkowicz [56] (1975) discussed a uniform cylinder that rolls without slipping down an incline on pp. 332-334 via angular momentum with respect to the point A of contact, claiming without discussion that the equation of motion is $\tau_A = I_A\,\alpha$, where α is the angular acceleration.

Desloge (1982) [58] discussed relative angular momentum $\mathbf{H}(a)$ with respect to a point a in his eq. (4), p. 227, but did not consider the corresponding torque equation. Rather, on p. 233 he remarked: *There are other point besides the origin and the center of mass for which*

[25] It may be that the intent was to consider the relative angular momentum $\mathbf{L}'_P$, but the wording used was inadvertently appropriate for the absolute angular momentum $\mathbf{L}_P$.

$\mathbf{G}(a)$ *(the torque* $\boldsymbol{\tau}_a$*) is equal to* $\dot{\mathbf{H}}(a)$ *(=* $d\mathbf{L}'_P/dt$*). ... It is true for an accelerating point if the acceleration of the point is directed toward the center or mass. .... However, the reader is advised to forget this special case. It is never necessary to use it, and the convenience that might be gained by retaining this case in one's repertoire is outweighed by the possibility of error it encourages.*

Faucher (1983) [59] gave a discussion of torque analyses for both absolute and relative angular momentum, $\mathbf{L}_{Aa}$ and $\mathbf{L}_A$, respectively, with respect to a point A (warning of possible confusion between them), following [54], which latter was no doubt unfamiliar to the physics community.

Greenwood (1988) discussed relative angular momentum on pp. 142-143 of [60], where it was called $\mathbf{H}_p$. Our equation of motion (15) appeared as eq. (4-70).

Crawford [61] (1989) gave a purposefully misleading torque analysis to provoke the reader into greater awareness of subtleties therein. This led to a series of comments [62]-[66].

Zypman [64] (1989) discussed relative angular momentum $\mathbf{L}_O$ with respect to a point O, where our equation of motion (15) appeared as eq. (4).

Illarremendi and Gaztelurrutia [66] (1995) discussed torque analyses based on both relative and absolute angular momenta $\mathbf{L}_P^A$ and $\mathbf{L}_P^R$, respectively, with respect to a point P in sec. 2, and included several references to relevant works in Spanish. They introduced the instantaneous center of rotation (ICR) on p. 251, and restricted discussion of it to when it is also the point of contact of the rigid body with a fixed surface (ignoring, for example, the case of rotation about a fixed axis). Perhaps because of the somewhat counterintuitive form of our eq. (36) for the case when the instantaneous center is regard as being in the rigid body (for which they cite Tiersten [65]), they emphasized the case that the instantaneous center is on the fixed surface in sec. 3.2, p. 253. As they remarked on p. 254: *Let us end pointing out that although it is correct to...take torques...about the ICR, we must be very careful.*

Baruh [67] (1999) defined the absolute angular momentum $\mathbf{H}_B$ about a moving point B in his eq. [3.3.6], p. 158, and gave the torque equation for this in eq. [3.3.15], but did not use it for any moving point except the center of mass G. In sec. 3.11, p, 200, he mentioned the instantaneous center C of rotation, but only gave an example of a rigid body with a symmetry axis that rolls without slipping, in which case the torque equation is $I_C\,\dot{\omega} = \tau_C$, unlike the general form of our eq. (36).

Knudsen and **Hjorth** [68] (2000) discussed torque analysis based on absolute angular momentum $\mathbf{L}_Q$ with respect to a point Q in sec. 11.8, p. 256. Our equation of motion (13) appeared at the bottom of p. 257. In example 12.5, p. 289 they discussed rolling without slipping of a cylinder, and mentioned that the point C of contact is the instantaneous axis of rotation, but analyzed this problem with respect to the center of mass.

Rodríguez [69] (2003) discussed relative angular momentum $\mathbf{L}_A$ with respect to a point A, where our equation of motion (15) appeared as eq. (1).

Morin [70] (2007) gave a limerick on p. 313 about our eq. (16), the "orbital" and "spin" decomposition of kinetic energy, which he called E:

To calculate E, my dear class,
Just add up two thing, and you'll pass.
Take the CM's point E,
And then add on with glee,
The E 'round the center of mass.

Then, in sec. 8.4.3, pp. 324-325, he discussed the relative angular momentum about a moving point, and the associated torque equation.

Theron [71] (2009) discussed the torque analysis for a rigid body with a symmetry plane, our eq. (36), and recalled Loney [35], Lambe [46] and Tiersten [65] as past advocates thereof. However, Theron may have been overoptimistic in his statement, p. 919: *There are no problems or subtle pitfalls when using moments around the instantaneous center in the various energy principles.*

Turner and Turner [72] (2010) discussed torque analyses, about the point P of contact of a rigid body that rolls without slipping on a fixed surface, without much awareness of the previous literature on this topic (such as Theron's article [71] published one year earlier in the same journal). They first claimed (p. 905) that torque equation is $I_P\,\dot{\omega} = \tau_P$ rather than $d(I_P\,\omega)/dt = I_P\,\dot{\omega} + \dot{I}_P\,\omega/2 = \tau_P$ as in our eq. (36) (since the point of contact is also the instantaneous center of rotation), even though they explicitly noted examples where I_P is time dependent. Rather than identifying the "missing" term $\dot{I}_P\,\omega/2$ as associated with the time-dependent moment of inertia, they instead invoked the relative angular momentum, our $\mathbf{L}'_P$, and wrote their eq. (14) which corresponds to the torque equation for relative angular momentum, our eq. (15), and then called the term $-(\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\mathbf{a}_{\rm cm}$ a "phantom torque". The present author finds this argument somewhat misleading, as the examples of [72] are well analyzed by the torque equation $d\mathbf{L}_P/dt = \boldsymbol{\tau}_P$, and the "error" was in the computation of the rate of change of angular momentum $d\mathbf{L}_P/dt$, rather than of the torque.

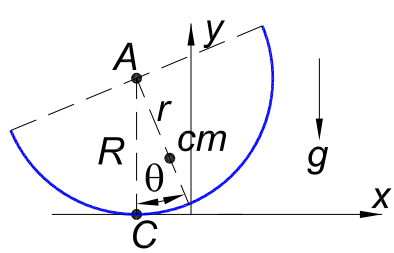
Jensen elaborated on the discussion of Turner and Turner in [73] (2011) and [75] (2012), noting that the point of contact of a rigid body with a fixed surface can be regarded as being in the rigid body (as in sec. 4.3 above), or on the fixed surface (as in [66], which was not cited).

Hu [74] (2011) made a different comment on the argument of Turner and Turner [72], that their examples can also be well treated by regarding the point of contact as fixed in the lab (sec. 4.3.1 above).

Examples of torque analyses by the present author include [86]-[96].

A Appendix: Rolling of a Half-Cylindrical Shell

As an illustration of various torque analyses, we consider a half-cylindrical shell of mass m and radius R that rolls without slipping on a horizontal plane. This problem was considered on p. 280 of [51],[26] and in [72].[27]



[26]The case of no friction at the horizontal plane, where the center of mass only moves vertically, was also considered in [51].

[27]An equivalent example was discussed on pp. 244-246 of [43].

The shell has two symmetry planes, but no symmetry axis that contains the point of contact C of the shell with the horizontal plane. We work with various points in the symmetry plane shown in the figure above, which plane is perpendicular to the (parallel) lines of the cylindrical surface (*i.e.*, this symmetry plane is the plane of the paper).[28]

The moment of inertia about the center A is $I_A = mR^2$, the distance from A to the center of mass is $r = 2R/\pi$, so by the parallel-axis theorem, the moments of inertia about the center of mass, and about the point C of contact with the plane when the shell has rotated by angle θ from its equilibrium position as shown in the sketch above, are,

$$I_A = mR^2, \qquad I_{\rm cm} = I_A - mr^2 = m(R^2 - r^2), \tag{39}$$

$$I_C = I_{\rm cm} + m(R^2 - 2rR\cos\theta + r^2) = 2m(R^2 - rR\cos\theta). \tag{40}$$

A.1 Lagrange's Method

We take the single generalized coordinate to be the angle θ. The coordinates of the center of mass are,

$$x_{\rm cm} = -R\theta + r\sin\theta, \qquad \dot{x}_{\rm cm} = -R\,\dot{\theta} + r\,\dot{\theta}\cos\theta, \qquad \ddot{x}_{\rm cm} = -R\,\ddot{\theta} + r\,\ddot{\theta}\cos\theta - r\,\dot{\theta}^2\sin\theta, \tag{41}$$

$$y_{\rm cm} = R - r\cos\theta, \qquad \dot{y}_{\rm cm} = r\,\dot{\theta}\sin\theta, \qquad \ddot{y}_{\rm cm} = r\,\ddot{\theta}\sin\theta + r\,\dot{\theta}^2\cos\theta, \tag{42}$$

so the kinetic and potential energies are,

$$T = \frac{I_{\rm cm}\,\dot{\theta}^2}{2} + \frac{m\,\dot{\theta}^2}{2}(R^2 - 2rR\cos\theta + r^2) = m\,\dot{\theta}^2(R^2 - rR\cos\theta), \quad V = mg(R - r\cos\theta). \tag{43}$$

To deduce the equation of motion we can either consider the constant energy $E = T + V$, or the Lagrangian $\mathcal{L} = T - V$, and take the appropriate derivatives to find,

$$\left(1 - \frac{r\cos\theta}{R}\right)\ddot{\theta} = -\frac{r\sin\theta}{2R}\,\dot{\theta}^2 - \frac{rg\sin\theta}{2R^2}, \qquad \left(1 - \frac{2\cos\theta}{\pi}\right)\ddot{\theta} = -\frac{\sin\theta}{\pi}\,\dot{\theta}^2 - \frac{g\sin\theta}{\pi R}. \tag{44}$$

For small angles, the equation of motion simplifies to,

$$\left(1 - \frac{2}{\pi}\right)\ddot{\theta} \approx -\frac{g\,\theta}{\pi R}, \tag{45}$$

so the angular frequency Ω of small oscillations is,

$$\Omega = \sqrt{\frac{g}{(\pi - 2)R}}. \tag{46}$$

[28] The other symmetry plane contains the lines parallel to the cylindrical surface that pass through points A and the center of mass.

A.2 Torque Analysis about the Origin: $d\mathbf{L}/dt = \boldsymbol{\tau}$

The angular momentum about the origin is,

$$\mathbf{L} = \mathbf{L}_{\rm cm} + m\mathbf{x}_{\rm cm} \times \mathbf{v}_{\rm cm}, \qquad L = I_{\rm cm}\,\dot{\theta} + m\,\dot{\theta}[R^2 + r^2 - rR(\theta\sin\theta + 2\cos\theta)] \tag{47}$$

$$\frac{dL}{dt} = m\,\ddot{\theta}[2R^2 - rR(\theta\sin\theta + 2\cos\theta)] - mrR\,\dot{\theta}^2(\theta\cos\theta - \sin\theta). \tag{48}$$

To evaluate the torque about the origin, we need the equation of motion of the center of mass of the shell, in terms of gravity and the force components F_x and F_y which act at the point C of contact with the horizontal plane,

$$F_x = m\ddot{x}_{\rm cm} = m(-R\,\ddot{\theta} + r\,\ddot{\theta}\cos\theta - r\,\dot{\theta}^2\sin\theta), \quad F_y - mg = m\ddot{y}_{\rm cm} = m[r\,\ddot{\theta}\sin\theta + r\,\dot{\theta}^2\cos\theta]. \tag{49}$$

Using this, the torque about the origin is,

$$\begin{aligned}\tau &= -x_{\rm cm}mg + x_C F_y = mg(R\theta - r\sin\theta) - R\,\theta\{mg + m[r\,\ddot{\theta}\sin\theta + r\,\dot{\theta}^2\cos\theta]\} \\ &= -mgr\sin\theta - mrR\,\theta\,\ddot{\theta}\sin\theta - mrR\,\theta\,\dot{\theta}^2\cos\theta. \end{aligned}\tag{50}$$

Finally, using the torque equation about the origin, $dL/dt = \tau$, we obtain the equation of motion,

$$\ddot{\theta}(2R^2 - 2rR\cos\theta) + rR\,\dot{\theta}^2\sin\theta = -gr\sin\theta, \tag{51}$$

as previously found in eq. (44).

A.3 Torque Analysis about the Center of Mass: $d\mathbf{L}_{\rm cm}/dt = \boldsymbol{\tau}_{\rm cm}$

The angular momentum about the center of mass is,

$$L_{\rm cm} = I_{\rm cm}\,\dot{\theta}, \tag{52}$$

$$\frac{dL_{\rm cm}}{dt} = m\,\ddot{\theta}(R^2 - r^2). \tag{53}$$

The torque about the center of mass is,

$$\begin{aligned}\tau_{\rm cm} &= (y_{\rm cm} - y_C)F_x - (x_{\rm cm} - x_C)F_y \\ &= m(R - r\cos\theta)(-R\,\ddot{\theta} + r\,\ddot{\theta}\cos\theta - r\,\dot{\theta}^2\sin\theta) - r\sin\theta\{mg + m[r\,\ddot{\theta}\sin\theta + r\,\dot{\theta}^2\cos\theta]\} \\ &= -mgr\sin\theta - m\,\ddot{\theta}(R^2 + r^2 + 2rr\cos\theta) - mrR\,\dot{\theta}^2\sin\theta. \end{aligned}\tag{54}$$

Then, using the torque equation about the center of mass, $dL_{\rm cm}/dt = \tau_{\rm cm}$, we obtain the equation of motion,

$$\ddot{\theta}(2R^2 - 2rR\cos\theta) + rR\,\dot{\theta}^2\sin\theta = -gr\sin\theta, \tag{55}$$

as previously found in eq. (44).

A.4 Torque Analyses about the Moving Point A

Before considering torque analyses about the point C of contact between the rolling half-cylindrical shell and the horizontal plane, we consider one other point, A, which would be the center of the shell if it were a full cylinder.[29] This case will serve as a first illustration of the use of the absolute and relative angular momenta, $\mathbf{L}_A$ and $\mathbf{L}'_A$, introduced in secs. 3.1-2 above.

Point A has coordinates,

$$x_A = x_C = -R\,\theta \qquad \dot{x}_A = -R\,\dot{\theta}, \qquad \ddot{x}_A = -R\,\ddot{\theta}, \tag{56}$$

$$y_A = R, \qquad \dot{y}_A = 0 = \ddot{y}_A. \tag{57}$$

A.4.1 Use of the Absolute Angular Momentum $\mathbf{L}_A$

The absolute angular momentum about point A is, according to eq. (12),

$$\mathbf{L}_A = \mathbf{L}_{\rm cm} + (\mathbf{x}_{\rm cm} - \mathbf{x}_A) \times m\,\mathbf{v}_{\rm cm}, \tag{58}$$

$$L_A = I_{\rm cm}\,\dot{\theta} - mrR\,\dot{\theta}\cos\theta + mr^2\,\dot{\theta} = mR^2\,\dot{\theta} - mrR\,\dot{\theta}\cos\theta, \tag{59}$$

$$\frac{dL_A}{dt} = mR^2\ddot{\theta} - mrR\,\ddot{\theta}\cos\theta + mrR\,\dot{\theta}^2\sin\theta. \tag{60}$$

The torque about point A is,

$$\begin{aligned} \tau_A &= -(x_{\rm cm} - x_A)mg + (y_C - y_A)F_x \\ &= -mgr\sin\theta + mR(-R\,\ddot{\theta} + r\,\ddot{\theta}\cos\theta - r\,\dot{\theta}^2\sin\theta). \end{aligned} \tag{61}$$

However, the torque equation in this case is not simply $dL_A/dt = \tau_A$, but rather eq. (13),

$$\frac{d\mathbf{L}_A}{dt} = \boldsymbol{\tau}_A + m\,\mathbf{v}_{\rm cm} \times \mathbf{v}_A, \tag{62}$$

$$\frac{dL_A}{dt} = \tau_A + mrR\,\dot{\theta}^2\sin\theta, \tag{63}$$

$$\begin{aligned} &mR^2\ddot{\theta} - mrR\,\ddot{\theta}\cos\theta + mrR\,\dot{\theta}^2\sin\theta \\ &\qquad = -mgr\sin\theta + mR(-R\,\ddot{\theta} + r\,\ddot{\theta}\cos\theta - r\,\dot{\theta}^2\sin\theta) + mrR\,\dot{\theta}^2\sin\theta, \end{aligned} \tag{64}$$

$$\ddot{\theta}(2R^2 - 2rR\cos\theta) + rR\,\dot{\theta}^2\sin\theta = -gr\sin\theta, \tag{65}$$

as previously found in eq. (44).[30]

A.4.2 Use of the Relative Angular Momentum $\mathbf{L}'_A$

The relative angular momentum about point A is, according to eq. (14),

$$\mathbf{L}'_A = \mathbf{L}_{\rm cm} + (\mathbf{x}_{\rm cm} - \mathbf{x}_A) \times m\,(\mathbf{v}_{\rm cm} - \mathbf{v}_A) \tag{66}$$

$$L'_A = I_{\rm cm}\,\dot{\theta}mr^2\,\dot{\theta} = mR^2\,\dot{\theta} = I_A\,\dot{\theta}, \tag{67}$$

$$\frac{dL'_A}{dt} = mR^2\,\ddot{\theta} = I_A\,\ddot{\theta}. \tag{68}$$

[29] We do not consider here use of, say, the point B on the horizontal plane where the extended diameter of the half cylinder intersects it.

[30] The torque analysis in this case was successful in using dL_A/dt of eq. (60), which is the time derivative of eq. (59).

Again, the torque equation in this case is not simply $dL'_A/dt = \tau_A$, but rather eq. (15),

$$\frac{d\mathbf{L}'_A}{dt} = \boldsymbol{\tau}_A + (\mathbf{x}_{\rm cm} - \mathbf{x}_A) \times (-m\,\mathbf{a}_A) \tag{69}$$

$$\frac{dL'_A}{dt} = \tau_A + mrR\,\ddot{\theta}\cos\theta, \tag{70}$$

$$mR^2\ddot{\theta} = -mgr\sin\theta + mR(-R\,\ddot{\theta} + r\,\ddot{\theta}\cos\theta - r\,\dot{\theta}^2\sin\theta) + mrR\,\ddot{\theta}\cos\theta, \tag{71}$$

$$\ddot{\theta}(2R^2 - 2rR\cos\theta) + rR\,\dot{\theta}^2\sin\theta = -gr\sin\theta, \tag{72}$$

as previously found in eq. (44).[31]

A.5 Torque Analyses about the Point C of Contact

We now consider five analyses using, in different ways, the Point C of contact between the half cylinder and the horizontal plane.[32]

We could consider is that the point C is fixed in the lab frame, and happens to be at the point of contact at the time of interest, as discussed in sec. 4.3.1. For fixed points of reference, there is no distinction between absolute and relative angular momentum.

We could also considering point C as moving along on the horizontal plane, always being at the point of contact between the rigid body and the fixed surface. In this case it has coordinates and time derivatives,

$$x_C = -R\,\theta \qquad \dot{x}_C = -R\,\dot{\theta}, \qquad \ddot{x}_C = -R\,\ddot{\theta}, \tag{73}$$

$$y_C = 0, \qquad \dot{y}_C = 0 = \ddot{y}_C. \tag{74}$$

In this view, the point C is not instantaneously at rest, and is not the instantaneous center of rotation. As point C is moving, we should consider both the absolute and relative angular momenta with respect to this point.

In addition, we could consider point C as being a fixed point in the rigid body that happens to be in contact with the horizontal surface at the time of interest, when it is instantaneously at rest. This version of point C has coordinates and time derivatives,

$$x^\star_C = -R\,\theta \qquad \dot{x}^\star_C = 0 = \ddot{x}^\star_C, \tag{75}$$

$$y^\star_C = 0 = \dot{y}^\star_C, \qquad \ddot{y}^\star_C = R\,\dot{\theta}^2. \tag{76}$$

In this view, point C is the instantaneous center of rotation. Again, as point C is moving, we should consider both the absolute and relative angular momenta with respect to this point.

Hence, we have five variants of torque analyses to consider with respect to (different conceptions of) the point C of contact.

[31] The torque analysis in this case was successful in using dL'_A/dt of eq. (68), which is the time derivative of eq. (67).

[32] These five analyses correspond to those in Table I of [75].

A.5.1 Point C is Fixed

As noted in sec. 4.3.1, for a fixed point C we can write,

$$\boldsymbol{\tau}_C = \frac{d\mathbf{L}_C}{dt} = \frac{d\mathbf{L}_{\rm cm}}{dt} + (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\,\mathbf{a}_{\rm cm}, \tag{77}$$

$$\frac{d\mathbf{L}_{\rm cm}}{dt} = \boldsymbol{\tau}_C + (\mathbf{x}_{\rm cm} - \mathbf{x}_C) \times (-m\,\mathbf{a}_{\rm cm}) = \boldsymbol{\tau}_{\rm cm}, \tag{78}$$

so this torque analysis is not operationally distinct from the analysis with respect to the center of mass, despite the different words used to introduce it.

For the present example, this analysis is equivalent to that given in sec. A.3, which led to the equation of motion as previously found in eq. (44).

A.5.2 Use of the Absolute Angular Momentum $\mathbf{L}_C$ for Point C Moving on the Plane

The absolute angular momentum about point C is, according to eq. (12),

$$\mathbf{L}_C = \mathbf{L}_{\rm cm} + (\mathbf{x}_{\rm cm} - \mathbf{x}_C) \times m\,\mathbf{v}_{\rm cm}, \tag{79}$$

$$L_C = I_{\rm cm}\,\dot\theta + m\,\dot\theta(R^2 + r^2 - 2rR\cos\theta) = I_C\,\dot\theta = 2m\,\dot\theta(R^2 - rR\cos\theta) \tag{80}$$

$$\frac{dL_C}{dt} = 2m\,\ddot\theta(R^2 - rR\cos\theta) + 2mrR\,\dot\theta^2\sin\theta, \tag{81}$$

recalling eq. (40) for I_C. The torque about point C is,

$$\tau_C = -(x_{\rm cm} - x_C)mg = -mgr\sin\theta. \tag{82}$$

However, the torque equation in this case is not simply $dL_C/dt = \tau_C$, but rather eq. (13),

$$\frac{d\mathbf{L}_C}{dt} = \boldsymbol{\tau}_C + m\,\mathbf{v}_{\rm cm} \times \mathbf{v}_C, \tag{83}$$

$$\frac{dL_C}{dt} = \tau_C + mrR\,\dot\theta^2\sin\theta, \tag{84}$$

$$2m\,\ddot\theta(R^2 - rR\cos\theta) + 2mrR\,\dot\theta^2\sin\theta = -mgr\sin\theta + mrR\,\dot\theta^2\sin\theta, \tag{85}$$

$$\ddot\theta(2R^2 - 2rR\cos\theta) + rR\,\dot\theta^2\sin\theta = -gr\sin\theta, \tag{86}$$

as previously found in eq. (44).

A.5.3 Use of the Absolute Angular Momentum $\mathbf{L}_C$ for Point C Moving with the Rigid Body

We now regard point C, at $\mathbf{x}_C^\star$, as moving with the rigid body, being instantaneously at rest at the time of interest.

The absolute angular momentum about point C is, according to eq. (12),

$$\mathbf{L}_C = \mathbf{L}_{\rm cm} + (\mathbf{x}_{\rm cm} - \mathbf{x}_C^\star) \times m\,\mathbf{v}_{\rm cm}, \tag{87}$$

$$L_C = I_{\rm cm}\,\dot\theta + m\,\dot\theta(R^2 + r^2 - 2rR\cos\theta) = I_C\,\dot\theta = 2m\,\dot\theta(R^2 - rR\cos\theta), \tag{88}$$

which is that same as found in eq. (80).

However, **if we took the time derivative of eq. (88) to get eq. (81), that would be wrong for the case of point C moving with the rigid body**, although that step was valid for point C moving on the fixed surface.

Rather, we should take the time derivative of eq. (87), noting the difference between $\mathbf{x}_C$ and $\mathbf{x}_C^\star$,

$$\frac{d\mathbf{L}_C}{dt} = \frac{d\mathbf{L}_{\rm cm}}{dt} - \mathbf{v}_C^\star \times m\,\mathbf{v}_{\rm cm} + (\mathbf{x}_{\rm cm} - \mathbf{x}_C^\star) \times m\,\mathbf{a}_{\rm cm} \tag{89}$$

$$\begin{aligned}\frac{dL_C}{dt} &= m\,\ddot\theta(R^2 - r^2) + mrR\,\dot\theta^2 \sin\theta + m\,\ddot\theta(R^2 + r^2 - 2rR\cos\theta) + mrR\,\dot\theta^2 \sin\theta \\ &= 2m\,\ddot\theta(R^2 - rR\cos\theta) + mrR\,\dot\theta^2 \sin\theta. \end{aligned}\tag{90}$$

Equivalently, we could recall eq. (36), that the time derivative of the absolute angular momentum about the instantaneous center of rotation is,

$$\frac{dL_C}{dt} = I_C\,\dot\omega + \frac{1}{2}\dot I_C\,\omega = 2m\,\ddot\theta(R^2 - rR\cos\theta) + mrR\,\dot\theta^2 \sin\theta. \tag{91}$$

Since point C is now the instantaneous center of rotation, the torque equation is just $dL_C/dt = \tau_C$,

$$2m\,\ddot\theta(R^2 - rR\cos\theta) + mrR\,\dot\theta^2 \sin\theta = -mgr\sin\theta \tag{92}$$

as previously found in eq. (44).

A.5.4 Use of the Relative Angular Momentum $\mathbf{L}'_C$ for Point C Moving on the Plane

The relative angular momentum about point C is, according to eq. (14),

$$\mathbf{L}'_C = \mathbf{L}_{\rm cm} + (\mathbf{x}_{\rm cm} - \mathbf{x}_C) \times m\,(\mathbf{v}_{\rm cm} - \mathbf{v}_C), \tag{93}$$

$$L'_C = I_{\rm cm}\,\dot\theta + mr^2\,\dot\theta - mrR\,\dot\theta\cos\theta = mR^2\,\dot\theta - mrR\,\dot\theta\cos\theta, \tag{94}$$

$$\frac{dL'_C}{dt} = mR^2\,\ddot\theta - mrR\,\ddot\theta\cos\theta + mrR\,\dot\theta^2\sin\theta. \tag{95}$$

The torque about point C is again,

$$\tau_C = -(x_{\rm cm} - x_C)mg = -mgr\sin\theta. \tag{96}$$

The torque equation in this case is not simply $dL'_C/dt = \tau_C$, but rather eq. (15),

$$\frac{d\mathbf{L}'_C}{dt} = \boldsymbol{\tau}_C + (\mathbf{x}_{\rm cm} - \mathbf{x}_C) \times (-m\,\mathbf{a}_C), \tag{97}$$

$$\frac{dL'_C}{dt} = \tau_C - mR\,\ddot\theta(R - r\cos\theta), \tag{98}$$

$$mR^2\,\ddot\theta - mrR\,\ddot\theta\cos\theta + mrR\,\dot\theta^2\sin\theta = -mgr\sin\theta - mR\,\ddot\theta(R - r\cos\theta), \tag{99}$$

$$\ddot\theta(2R^2 - 2rR\cos\theta) + rR\,\dot\theta^2\sin\theta = -gr\sin\theta, \tag{100}$$

as previously found in eq. (44).

A.5.5 Use of the Relative Angular Momentum $\mathbf{L}'_C$ for Point C Moving with the Rigid Body

The relative angular momentum about point C, now at $\mathbf{x}^\star_C$, is, according to eq. (14),

$$\mathbf{L}'_C = \mathbf{L}_{\rm cm} + (\mathbf{x}_{\rm cm} - \mathbf{x}^\star_C) \times m\,(\mathbf{v}_{\rm cm} - \mathbf{v}^\star_C), \tag{101}$$

$$L'_C = I_{\rm cm}\,\dot\theta + mR^2\,\dot\theta + mr^2\,\dot\theta - 2mrR\,\dot\theta\cos\theta = 2mR^2\,\dot\theta - 2mrR\,\dot\theta\cos\theta, \tag{102}$$

However, as in sec. A.5.3, **if we now take the time derivative of eq. (102) we get the "wrong" value for dL'_C/dt.**

Again, we should take the time derivative of the more basic expression (101),

$$\frac{d\mathbf{L}'_C}{dt} = \frac{d\mathbf{L}_{\rm cm}}{dt} + (\mathbf{x}_{\rm cm} - \mathbf{x}^\star_C) \times m\,(\mathbf{a}_{\rm cm} - \mathbf{a}^\star_C), \tag{103}$$

$$\frac{dL'_C}{dt} = I_{\rm cm}\,\ddot\theta + mR^2\,\ddot\theta + mr^2\,\ddot\theta - 2mrR\,\ddot\theta\cos\theta = 2mR^2\,\ddot\theta - 2mrR\,\ddot\theta\cos\theta, \tag{104}$$

The torque about point C is again,

$$\tau_C = -(x_{\rm cm} - x_C)mg = -mgr\sin\theta. \tag{105}$$

The torque equation in this case is not simply $dL'_C/dt = \tau_C$, but rather eq. (15),

$$\frac{d\mathbf{L}'_C}{dt} = \boldsymbol{\tau}_C + (\mathbf{x}_{\rm cm} - \mathbf{x}^\star_C) \times (-m\,\mathbf{a}^\star_C), \tag{106}$$

$$\frac{dL'_C}{dt} = \tau_C - mrR\,\dot\theta^2\sin\theta, \tag{107}$$

$$2mR^2\,\ddot\theta - 2mrR\,\ddot\theta\cos\theta = -mgr\sin\theta - mrR\,\dot\theta^2\sin\theta, \tag{108}$$

$$\ddot\theta(2R^2 - 2rR\cos\theta) + rR\,\dot\theta^2\sin\theta = -gr\sin\theta, \tag{109}$$

as previously found in eq. (44).

A.6 Comments

Of the ten analyses presented above for the example of a half-cylindrical shell rolling without slipping on a horizontal surface, most people would consider Lagrange's method to be the simplest (and most elegant). Yet, there remains some desire in many of us to "wallow in the Newtonian mud", perhaps because of the good feeling one gets on making a successful torque analysis. However, these longer analyses offer many opportunities for error. This appears especially true for the analyses using absolute or relative angular momentum about the instantaneous center of rotation (= point of contact between the rolling body and the fixed surface) when the rolling body does not have a symmetry axis, as in these cases $d\mathbf{L}_C/dt \neq \partial\mathbf{L}_C/\partial t$ (and $d\mathbf{L}'_C/dt \neq \partial\mathbf{L}'_C/\partial t$).

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