The Shape of Tokamak Coils

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1 Problem

The toroidal magnets used in magnetic confinement fusion devices such as a tokamak consist of n coils each carrying a large DC current I, arrayed around a common z axis. Deduce the shape $z(r)$, in cylindrical coordinates (r, θ, z), of the coils such that the magnetic force on each coil is balanced by the mechanical tension, which latter is uniform along the coil. In this case, the magnetic force will not bend the coils (although it does try to expand them).

2 Solution

The azimuthal magnetic field of the toroid follows from Ampère's Law,

$$B_\theta(r) = \frac{2nI}{rc}\,, \tag{1}$$

in Gaussian units, assuming n is large. The magnetic force on a segment of a coil of length $dl = \sqrt{1 + z'^2}\,dr$ is,

$$d\mathbf{F}_{\rm mag} = \frac{I\,dl}{c}\mathbf{B} \times \hat{\mathbf{i}} = \frac{2nI^2B\,dr}{c^2 r}(-z', 0, 1), \tag{2}$$

where $z' = dz/dr$, and the unit vector $\hat{\mathbf{i}}$ along the direction of the current is,

$$\hat{\mathbf{i}} = \frac{(1, 0, z')}{\sqrt{1 + z'^2}}\,. \tag{3}$$

The mechanical tension T in the coils leads to a force on the segment given by,

$$d\mathbf{F}_{\rm mech} = T\left[\frac{(1, 0, z'(r + dr))}{\sqrt{1 + z'^2(r + dr)}} - \frac{(1, 0, z'(r))}{\sqrt{1 + z'^2(r)}}\right] = \frac{Tz''\,dr}{(1 + z'^2)^{3/2}}(-z', 0, 1). \tag{4}$$

We desire the total force to be zero, which implies that,

$$\frac{z''}{(1 + z'^2)^{3/2}} = -\frac{2nI^2B}{c^2Tr}\,. \tag{5}$$

The first integral of this is,

$$\frac{z'}{\sqrt{1 + z'^2}} = -\frac{2nI^2B}{c^2T}\ln\frac{r}{r_0} \equiv -K\ln\frac{r}{r_0}\,, \qquad z' = -\frac{K\ln(r/r_0)}{\sqrt{1 - K^2\ln^2(r/r_0)}}\,, \tag{6}$$

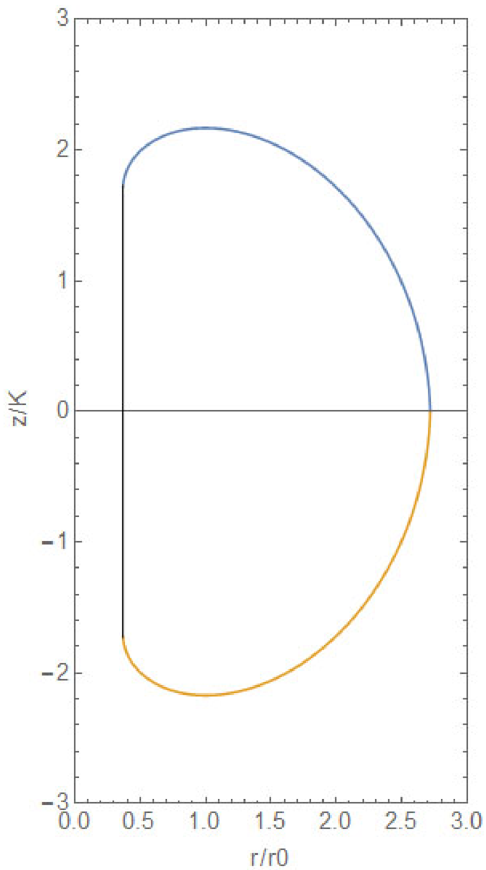
where $z'(r_0) = 0$ and $K = 2nI^2B/c^2$. At the minimum and maximum radii, the slope of the coil goes to $\pm\infty$, so $z'/\sqrt{1+z'^2} \to \pm 1$. Thus, the extreme radii are given by,

$$r_{\rm max,min} = r_0 e^{\pm 1/K}. \tag{7}$$

We take $z(r_{\rm max}) = 0$, such that,

$$z(r) = \int_{r_{\rm max}}^{r} z'\,dr = \int_{r}^{r_{\rm max}} dr\,\frac{K\ln(r/r_0)}{\sqrt{1 - K^2\ln^2(r/r_0)}}\,, \tag{8}$$

for $r_{\rm min} < r < r_{\rm max}$.[1]



[1]The figure was generated via the Mathematica notebook http://kirkmcd.princeton.edu/examples/tokamak.nb.