Electron Trajectories in a Hall Thruster

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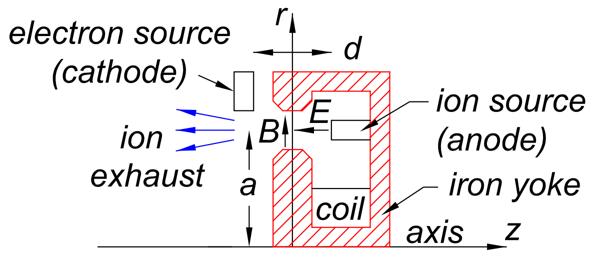
1 Problem

Discuss the electron trajectories in a Hall thruster, which device is sketched below. Hall thrusters are used on spacecraft to provide small correcting torques.[1]



An annular gap of central radius $r = a$ in the iron yoke of an axially symmetric electromagnet leads to a radial magnetic field in the gap. An axial electric field is also maintained in the gap. Show that electrons can move in circular orbits of radius $r_0 \approx a$ for suitable electric and magnetic field strengths, and that the orbital velocity is approximately equal to the so-called $\mathbf{E} \times \mathbf{B}$ drift velocity associated with uniform, crossed electric and magnetic fields.[2] Show also that for small perturbations about the ideal circular orbit lead to oscillatory motion at the local cyclotron frequency, with radial oscillations that are small compared to the axial.

The role of the orbiting electrons is to ionize gas atoms emitted from the cathode, such that the positive ions are then accelerated through the electric field, leading to a propulsive reaction force in the $+z$ direction. Discuss the small deflection of the heavy ions by the electromagnetic fields, noting that the canonical angular momentum is conserved.

A "Hall-effect" thruster is only indirectly related to the usual Hall effect. The terminology may be due to the fact that an axial current in a plasma would result in a radial $\mathbf{J} \times \mathbf{B}$ *pinch due to the self field of the current. The resulting radial current, if any, could be called a Hall-effect current, and the* $\mathbf{J} \times \mathbf{B}$ *force on the radial current points in the axial direction, providing an axial thrust on the system [3]. However, this geometry is not very practical for a thruster, and in the 1960's the configuration sketched in the figure above emerged as a more viable candidate for use on spacecraft. See, for example, [4]. Deployment of these devices is particularly due to the efforts of Morozov et al. [5]. A full understanding of the operation of these complex devices goes well beyond the scope of this example, and is the topic of ongoing research. See, for example, [6].*

Thrusters based on other configurations of crossed electric and magnetic fields have also been considered. For a pedagogic discussion of the "Lorentz force accelerator" (magnetoplasmodynamic thruster), see [7].

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[1] See, for example, http://htx.pppl.gov/

[2] See, for example, sec. 22 of [1] and sec. 12.3 of [2].

2 Solution

2.1 Circular Orbits for Electrons

We suppose that in the annular region between the iron pole tips the magnetic field can be approximated as purely radial,

$$\mathbf{B} = B_r\,\hat{\mathbf{r}} = B_0\frac{a}{r}\,\hat{\mathbf{r}}, \tag{1}$$

in a cylindrical coordinate system (r, ϕ, z), where B_0 is the magnetic field strength at radius a of the center of the annulus. A vector potential for the field (1) can be written,

$$\mathbf{A} = -B_r z\,\hat{\boldsymbol{\phi}} = -B_0\frac{az}{r}\,\hat{\boldsymbol{\phi}}, \tag{2}$$

where the axial extent of the annulus is centered on $z = 0$. The electric field in the annulus is approximately uniform and dominantly in the $-z$ direction,

$$\mathbf{E} \approx -E_0\,\hat{\mathbf{z}}, \tag{3}$$

which can be deduced from a scalar potential,

$$V \approx E_0 z. \tag{4}$$

It is desired that electrons move in circular orbits at constant r and z within the annulus, which requires that the axial force due to the electric field be canceled by the $\mathbf{v}\times\mathbf{B}$ Lorentz force. In addition, there must be a centripetal force, which can only be caused by a radial component of the electric field. Hence, we look beyond the approximations (3)-(4) and consider a simple form for an electric field with azimuthal symmetry that points largely in the $-z$ direction. A suitable scalar potential is,

$$V = E_0\frac{a(z + d/2)}{r}\,, \tag{5}$$

where the anode is at $z = d/2$, the cathode is at $z = -d/2$, and the voltage difference is $V_0 = E_0 d$. The corresponding electric field is,

$$\mathbf{E} = E_0\frac{a(z + d/2)}{r^2}\,\hat{\mathbf{r}} - E_0\frac{a}{r}\,\hat{\mathbf{z}}. \tag{6}$$

The (nonrelativistic) equations of motion for an electron of charge $-e$ and mass m in the fields (2) and (5) are,

$$m\ddot{r} - mr\dot{\phi}^2 = -eE_r = -eE_0\frac{a(z + d/2)}{r^2}\,, \tag{7}$$

$$m\frac{d}{dt}(r\dot{\phi}) + m\dot{r}\dot{\phi} = \frac{m}{r}\frac{d}{dt}(r^2\dot{\phi}) = -e\frac{\dot{z}B_r}{c} = -\frac{ea\dot{z}B_0}{cr}\,, \tag{8}$$

$$m\ddot{z} = -eE_z + e\frac{r\dot{\phi}B_r}{c} = eE_0 + \frac{ea\dot{\phi}B_0}{c}\,. \tag{9}$$

One solution to these equation is uniform circular motion with radius r_0 and angular velocity $-\Omega\,\hat{\boldsymbol{\phi}}$ in the plane z_0, where,

$$\Omega^2 = eE_0 \frac{a(z_0 + d/2)}{m r_0^3}\,, \tag{10}$$

from eq. (7), and,

$$\Omega = \frac{cE_0}{aB_0} \tag{11}$$

according to eq. (23). Together these imply that the parameters r_0 and z_0 of the circular orbit are related to the field strengths by,

$$\frac{eE_0 a(z_0 + d/2)}{mc^2 r_0} = \frac{E_0^2}{B_0^2}\,. \tag{12}$$

We are interested in orbits with $r_0 \approx a$ and $z_0 \approx 0$, so the magnetic field strength should be related to that of the electric field by,

$$B_0 \approx E_0 \sqrt{\frac{2mc^2}{eE_0 d}} = \frac{V_0}{d}\sqrt{\frac{2mc^2}{eV_0}} \gg E_0. \tag{13}$$

In SI units eq. (13) is,

$$B_0 \approx \frac{V_0}{cd}\sqrt{\frac{2mc^2}{eV_0}}\,. \tag{14}$$

For example, if $V_0 = 1000$ *V and* $d = 0.1$ *m, then* $B_0 \approx 10$ *Gauss.*

The kinetic energy of the orbiting electron is,

$$\frac{1}{2} m r_0^2 \Omega^2 \approx \frac{1}{2} m a^2 \Omega^2 \approx \frac{eV_0}{2}\,, \tag{15}$$

using eqs. (11) and (13), which is equal to the energy gained by the electron in moving from the cathode to $z_0 \approx 0$. Electrons from the cathode can assume the circular orbit only if they are scattered into it, in which case they can also be scattered out of it. Hence, there is a relatively narrow range of background gas densities over which electrons have both a reasonable probability of being captured into orbit, and remaining in orbit for several revolutions.

In general, the electrons will scatter into a perturbed orbit, which we consider below.

2.2 Electron Trajectories with an Axial Perturbation

Equation (8) integrates to,

$$m r^2 \dot{\phi} + \frac{eazB_0}{c} = \text{constant} = r\left(m r \dot{\phi} - \frac{eA_\phi}{c}\right) = L_z, \tag{16}$$

indicating that the z component of (canonical) angular momentum is conserved, as holds because the fields are independent of azimuth. Using this, eq. (9) can be rewritten as,

$$\ddot{z} + \left(\frac{eaB_0}{mcr}\right)^2 z = \frac{eE_0}{m} + \frac{eaB_0 L_z}{m^2 c r^2} \tag{17}$$

This suggests that we consider trajectories with constant (or nearly constant) radius r_0 and oscillatory axial motion,

$$z = z_0 + \epsilon r_0 \cos \omega t, \tag{18}$$

where,

$$z_0 = \frac{c}{eaB_0}\left(L_z + \frac{mcr_0^2 E_0}{aB_0}\right), \tag{19}$$

and the angular frequency ω of oscillation is, recalling eq. (11),

$$\omega = \frac{eaB_0}{mcr_0} \approx \frac{eB_0}{mc} \approx \frac{2a}{d}\Omega, \tag{20}$$

which is equal to the cyclotron frequency of circular orbits of electrons in a constant magnetic field of strength aB_0/r_0. For trajectories with $z_0 \approx 0$, the canonical angular momentum is,

$$L_z \approx -\frac{mcr_0^2 E_0}{aB_0} = -mr_0^2\Omega. \tag{21}$$

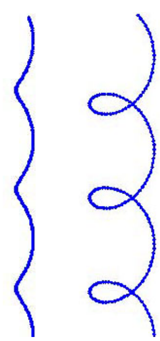
The azimuthal angular velocity $\dot\phi$ follows from eqs. (16) and (21) as,

$$\dot\phi \approx -\Omega - \frac{\omega z}{r_0} \approx -\Omega - \epsilon\omega \cos \omega t. \tag{22}$$

Hence, the azimuthal position $s = r_0\phi$ obeys,

$$s \approx -r_0(\Omega t + \epsilon \sin \omega t). \tag{23}$$

Representative electron trajectories on the cylinder of radius r_0 are sketched below for $\epsilon\omega$ less than, or greater than, Ω.



2.3 Electron Trajectories with Axial and Radial Perturbations

In general, the electron trajectories with no have constant r, so we consider perturbations about the ideal circular trajectory of radius r_0 and $z \approx 0$ with $L_z \approx -mr_0^2\Omega$ of the form,

$$r = r_0(1 + \delta \cos \omega t), \tag{24}$$

$$\dot\phi = -\Omega - \gamma\omega \cos \omega t, \tag{25}$$

$$z = \epsilon r_0 \cos \omega t. \tag{26}$$

Inserting these forms into eq. (16) and requiring the terms in $\cos\omega t$ to sum to zero, we find,

$$\delta = \frac{\omega}{2\Omega}(\epsilon - \gamma) \approx \frac{a}{d}(\epsilon - \gamma), \tag{27}$$

recalling eq. (20). Similarly, the radial equation (7) leads to the relation,

$$\delta \approx \frac{d}{2a}\frac{\epsilon - 2\gamma}{1 - d^2/4a^2} . \tag{28}$$

From this, we find,

$$\gamma \approx \epsilon\frac{1 - 3d^2/4a^2}{1 - 5d^2/4a^2} \approx \epsilon\left(1 - \frac{d^2}{2a^2}\right), \tag{29}$$

and,

$$\delta \approx \epsilon\frac{d}{2a} . \tag{30}$$

The radial excursions are small compared to the axial ones provided $d/2a \ll 1$, as holds in typical Hall thruster designs.

Using eqs. (24)-(26) in the axial equation (9) leads to the requirement that $\gamma = \epsilon$. This is satisfied to a good approximation according to eq. (29).

2.4 Deflection of the Positive Ions

The actual thrust of a Hall thruster is not due to the orbiting electrons, but to atoms ionized by these electrons. The reaction to the subsequent axial acceleration of the ions in the electric field forces the thrust in the $+z$ direction.

The heavy ions are little deflected by the magnetic field, but they do experience a small azimuthal kick as they exit the field. This kick is readily analyzed via the conserved canonical angular momentum (16), which for positive ions of charge e and mass M is,

$$L_z = r\left(Mr\dot{\phi} + \frac{eA_\phi}{c}\right) = Mrv_\phi + \frac{eazB_0}{c} , \tag{31}$$

recalling the vector potential (2) of the radial magnetic field (1).

An atom ionized at (r_0, ϕ_0, z_0) reaches the end of the magnetic field region ($z \approx -d/2$) with little change in its coordinates r of ϕ. However, radial magnetic field interacts with the axial velocity of the ion to deflect it azimthally. When the ion reaches the region where the vector potential is negligible, it has azimuthal velocity $v_{\phi,f}$ related by,

$$L_{z,i} = \frac{eaz_0B_0}{c} = L_{f,z} \approx Mr_0v_{\phi,f}, \tag{32}$$

so that,

$$v_{\phi,f} \approx= \frac{eaz_0B_0}{Mcr_0} \approx \frac{ez_0B_0}{Mc} = \omega_M z_0, \tag{33}$$

where $\omega_M = eB_0/Mc$ is the cyclotron frequency of the ion.

The sign of the azimuthal velocity (33) depends on where the atom was ionized. Clearly, the average azimuthal velocity of the exiting ions will be zero if the average position of creation of the ions is $z = 0$, *i.e.*, at the center of the radial field region.

References

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