# J.J. Thomson and "Hidden" Momentum

Kirk T. McDonald
*Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544*
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The term "hidden momentum" was introduced by Shockley [1] in 1967 to describe the small amount of net mechanical momentum that must exist in systems "at rest" that have nonzero electromagnetic field momentum.[1] The classic example is a current loop (Ampèrian magnetic dipole) in an external static electric field (perhaps due to a single electric charge) [4, 5].[2] This example had been considered by J.J. Thomson in 1904 [7, 8, 9], when he deduced via two different methods that the field momentum is (in Gaussian units),

$$\mathbf{P}_{\rm EM} = \frac{\mathbf{E}\times\mathbf{m}}{c}\,, \tag{1}$$

where the external electric field $\mathbf{E}$ is approximately uniform over the magnetic moment $\mathbf{m}$ (where $m = IA/c$ for a loop of area $A$ with current $I$), and $c$ is the speed of light in vacuum.[3,4]

In this note we review Thomson's various comments on electromagnetic field momentum, transcribing them from Maxwell's vector-component notation [11, 12] into contemporary usage.

## 1 Radiation Pressure and the Momentum of Light

Apparently, Kepler considered the pointing of comets' tails away from the Sun as evidence for radiation pressure of light [18]. After his unification of electricity, magnetism and light [11], Maxwell argued (sec. 792 of [12]) that the radiation pressure $P$ of light is equal to its energy density $u$,

$$P = u = \frac{D^2}{4\pi} = \frac{H^2}{4\pi} \tag{2}$$

for an electromagnetic wave with fields $\mathbf{D}$ and $\mathbf{H}$ in vacuum, but he did not explicitly associate this pressure with momentum in the electromagnetic field.[5]

Building on Faraday's electrotonic state,[6] Maxwell did have a conception of electromagnetic momentum, computed as [11, 12] (see also [17]),

$$\mathbf{P}_{\rm EM}^{\rm (Maxwell)} = \int \frac{\rho\mathbf{A}^{(C)}}{c}\,d{\rm Vol}, \tag{3}$$

[1] Shockley's notion was clarified in [2]. For a general definition of "hidden" momentum, see [3].

[2] For discussion of a recent misunderstanding of this example, see [6].

[3] This example illustrates that "hidden" momentum is an effect of order $1/c^2$, and hence can be called "relativistic". Thomson worked in the ESU and EMU systems, in which factors of $c$ do not appear in Maxwell's equations (nor in eq. (1)), so the "relativistic" aspects of his analyses were often not evident.

[4] The result (1) next appears in eq. (7.85) of [4], and then in [10], which latter cites Thomson [9] regarding Gilbertian magnetic monopoles but not for Ampèrian magnetic dipoles.

[5] Maxwell (and Thomson and Lorentz and most others influenced by the concept of a material aether), regarded the fields $\mathbf{D}$ and $\mathbf{H}$ as more "basic" than $\mathbf{E}$ and $\mathbf{B}$.

[6] Art. 60 of [13], Art. 1661 of [14], Arts. 1729 and 1733 of [15], and Art. 3269 of [16].

where $\rho$ is the electric charge density and $\mathbf{A}^{(C)}$ is the vector potential in the Coulomb gauge (that Maxwell used prior to the explicit recognition of gauge conditions [19]), but the form (3) seems to associate the momentum with charges rather than with fields. See also sec. 4.

In 1891, Thomson noted [20] that a sheet of electric displacement $\mathbf{D}$ (parallel to the surface) which moves perpendicular to its surface with velocity $\mathbf{v}$ must be accompanied by a sheet of magnetic field $\mathbf{H} = \mathbf{v}/c \times \mathbf{D}$ according to the free-space Maxwell equation $\boldsymbol{\nabla} \times \mathbf{H} = (1/c)\, \partial\mathbf{D}/\partial t$.[7] Then, the motion of the energy density of these sheets implies there is also a momentum density, eqs. (2) and (6) of [20],

$$\mathbf{p}_{\rm EM}^{\rm (Thomson)} = \frac{\mathbf{D} \times \mathbf{H}}{4\pi c}\,. \tag{4}$$

Also in 1891, Heaviside identified the momentum of the *free ether* in sec. 26 of [25] as,[8]

$$\mathbf{p}_{\rm EM}^{\rm (Heaviside)} = \frac{\mathbf{D} \times \mathbf{B}}{4\pi c}\,. \tag{5}$$

This was a clarification of his discussion in 1886, eq. (7a) of [28], of a *magnetoelectric force* $\mathbf{D}/4\pi c \times \partial\mathbf{B}/\partial t$.[9]

In 1893, Thomson transcribed much of his 1891 paper into the beginning of *Recent Researches* [31], adding the remark (p. 9) that the momentum density (4) is closely related to the Poynting vector [32, 33],[10,11]

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H}. \tag{6}$$

The form (4) was also used by Poincaré in 1900 [39], following Lorentz' convention [40] that the force on electric charge $q$ be written $q(\mathbf{D} + \mathbf{v}/c \times \mathbf{H})$, and that the Poynting vector be $(c/4\pi)\, \mathbf{D} \times \mathbf{H}$. In 1903 Abraham [41] argued for,

$$\mathbf{p}_{\rm EM}^{\rm (Abraham)} = \frac{\mathbf{E} \times \mathbf{H}}{4\pi c} = \frac{\mathbf{S}}{c^2}\,, \tag{7}$$

and in 1908 Minkowski [42] advocated the form,[12,13]

$$\mathbf{p}_{\rm EM}^{\rm (Minkowski)} = \frac{\mathbf{D} \times \mathbf{B}}{4\pi c}\,. \tag{8}$$

[7] Variants of Thomson's argument were given by Heaviside in 1891, sec. 45 of [21], and much later by Feynman in sec. 18-4 of [23], where it was noted that Faraday's law, $\boldsymbol{\nabla} \times \mathbf{E} = -(1/c)\, \partial\mathbf{B}/\partial t$, combined with the Maxwell equation for $\mathbf{H}$ implies that $v = c$ in vacuum, which point seems to have been initially overlooked by Thomson, although noted by him in sec. 265 of [24].

[8] See also p. 557 of [26] and p. 495 of [27].

[9] Heaviside also mentioned this concept in 1889 on pp. 399-330 of [30].

[10] Thomson argued, in effect, that the field momentum density (4) is related by $\mathbf{p}_{\rm EM} = \mathbf{S}/c^2 = u\mathbf{v}/c^2$ [20, 31]. See also eq. (19), p. 79 of [22], and p. 6 of [34]. The energy flow velocity defined by $\mathbf{v} = \mathbf{S}/u$ cannot exceed $c$; see, for example, sec. 2.1.4 of [35] and sec. 4.3 of [36].

[11] The idea that an energy flux vector is the product of energy density and energy flow velocity seems to be due to Umov [37], based on Euler's continuity equation [38] for mass flow, $\boldsymbol{\nabla} \cdot (\rho\mathbf{v}) = -\partial\rho/\partial t$.

[12] Minkowski, like Poynting [32], Heaviside [33] and Abraham [41], wrote the Poynting vector as $\mathbf{E} \times \mathbf{H}$. See eq. (75) of [42].

[13] For some remarks on the "perpetual" Abraham-Minkowski debate see [43].

Thomson did not relate the momentum density (4) to the radiation pressure $P$ of light, eq. (2), until 1904 (p. 355 of [8]) when he noted that $P = F/A = c\, p_{\rm EM} = D^2/4\pi = H^2/4\pi$ for fields moving with speed $c$ in vacuum, for which $D = H$. He also gave an argument (p. 348 of [8]) that the forms (3) and (4) for field momentum are equivalent once the sources of the fields are taken into account.[14]

# 2 Magnetic Pole plus Electric Charge

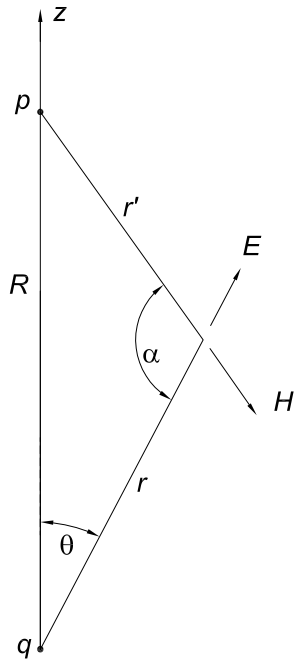
Thomson's 1904 paper [8] begins with considerations of a (Gilbertian) magnetic (mono)pole $p$ and electric charge $q$, both at rest.[15]

## 2.1 Field Momentum

Suppose the electric charge $q$ is at the origin, and the magnetic pole $p$ at distance $R$ away along the positive $z$-axis, as shown in the figure below. Then, the (Abraham) field-momentum density is, in spherical coordinates $(r, \theta\, \phi)$,

$$\mathbf{p}_{\rm EM} = \frac{\mathbf{E} \times \mathbf{H}}{4\pi c} = \frac{pq \sin\alpha}{4\pi c\, r^2 r'^2}\, \hat{\boldsymbol{\phi}} = \frac{pqR \sin\theta}{4\pi c\, r^2 r'^3}\, \hat{\boldsymbol{\phi}}\,, \tag{9}$$

noting that $H = p/r'^2$ for the magnetic pole, and that $\sin\alpha/R = \sin\theta/r'$ by the sine law.




[14] Possibly, Thomson delayed publishing the relation of radiation pressure to his expression (4) until he could demonstrate its equivalence to Maxwell's form (3). For other demonstrations of this equivalence, see Appendix B of [3], and [44].

[15] The motion of an electric charge with respect to a much heavier magnetic pole lies on the surface of a cone, as discussed in [45, 46].

The electromagnetic momentum (9) circulates azimuthally, such that the total electromagnetic momentum $\mathbf{P}_{\rm EM}$ is zero,

$$\mathbf{P}_{\rm EM} = \int \mathbf{p}_{\rm EM}\, d{\rm Vol} = 0. \tag{10}$$

Further, the total electromagnetic-field momentum for any configuration of static magnetic poles and electric charges is zero, being the sum of the momenta of all pairs of such particles. Hence, Thomson demonstrated the notable fact that **Electromagnetic field momentum can be nonzero only if electric charges, or Gilbertian magnetic poles (should they exist), are in motion.**[16]

## 2.2 Field Angular Momentum

Thomson also considered the angular momentum in the electromagnetic fields of the pole plus charge,

$$\mathbf{L}_{\rm EM} = \int \mathbf{r} \times \mathbf{p}_{\rm EM}\, d{\rm Vol} = -\frac{pqR}{4\pi c} \int \frac{r \sin\theta}{r^2 r'^3}\, d{\rm Vol}\, \hat{\boldsymbol{\theta}}. \tag{11}$$

This has only a nonzero $z$-component,

$$\begin{aligned} L_{{\rm EM},z} &= \frac{pqR}{2c} \int_{-1}^{1} \sin^2\theta\, d\cos\theta \int_0^\infty \frac{r\, dr}{(r^2 - 2rR\cos\theta + R^2)^{3/2}} \\ &= \frac{pqR}{2c} \int_{-1}^{1} \sin^2\theta\, d\cos\theta\, \frac{1+\cos\theta}{R\sin^2\theta} = \frac{pq}{c}\,, \end{aligned} \tag{12}$$

using Dwight 380.013. The angular momentum vector $\mathbf{L}_{\rm EM}$ points from the electric charge $q$ to the magnetic pole $p$.

In 1904 the notion of quantizing angular momentum was still years away, and the provocative result (12), that the angular momentum of a magnetic pole plus electric charge is independent of their separation, went unremarked until 1931 when Dirac [48] argued that $pq/c = \hbar/2$. See also sec. 6.12 of [49].[17]

# 3 Magnetic Dipole plus Electric Charge

According to the result of sec. 2.1, a Gilbertian magnetic dipole plus electric charge, all at rest, has zero total field momentum. Thomson then considered the nontrivial case of an Ampèrian magnetic dipole plus electric charge in two different models.

[16] The nonzero field momentum (1) is associated with a "static" current loop, which involves electric charges in motion. See also [47].

[17] The field angular momentum (12) had been deduced by Poincaré in 1896 [50, 51], but was not recognized as such at the time.

## 3.1 Ampèrian Magnetic Dipole as a Small Solenoidal Coil

On p. 347 of [8], Thomson noted that the external magnetic field of a Gilbertian magnetic dipole is the same as that of an Ampèrian dipole, so the field momentum of the latter (in the presence of an electric charge) is just the momentum associated with the "interior" of the dipole. If the magnetic dipole is realized by a coil of area $A$ and length $l$ with $N$ turns that carry current $I$, then the interior axial field is $H_{\rm in} \approx (4\pi/c)NI/l = (4\pi/c)NIA/{\rm Vol}_{\rm coil} = 4\pi m/{\rm Vol}_{\rm coil}$, where the magnetic moment of the coil is $m = NIA/c$. Hence, the field momentum inside the coil (and also the total field momentum of the system) is,[18]

$$\mathbf{P}_{\rm EM} \;=\; \frac{\mathbf{E}\times\mathbf{H}_{\rm in}{\rm Vol}_{\rm coil}}{4\pi c} \;=\frac{\mathbf{E}\times\mathbf{m}}{c}\,, \tag{13}$$

as in eq. (1), where $\mathbf{E}$ is the electric field of charge $q$ at the magnetic dipole $\mathbf{m}$.

## 3.2 Field Momentum via Maxwell's Relation (3)

An Ampèrian magnetic dipole is not necessarily well described as a small solenoid, so Thomson gave a second derivation of the field momentum for a magnetic dipole plus electric charge in sec. 285 of [9].[19] This was based on Maxwell's relation (3) for the field momentum, noting that the vector potential (in the Coulomb gauge) of an Ampèrian magnetic dipole $\mathbf{m}$ at the origin is,[20]

$$\mathbf{A}^{\rm (C)} = \frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}\,. \tag{14}$$

Then, the field momentum of the magnetic dipole plus electric charge $q$ is,

$$\mathbf{P}_{\rm EM} = \int \frac{\rho\mathbf{A}^{(C)}}{c}\,d{\rm Vol} = \frac{\mathbf{m}\times q\hat{\mathbf{r}}}{cr^2} = \frac{\mathbf{E}\times\mathbf{m}}{c}\,, \tag{15}$$

noting that the electric field $\mathbf{E}$ at the magnetic dipole $\mathbf{m}$ is $-q\,\hat{\mathbf{r}}/r^2$, as $\hat{\mathbf{r}}$ points from $\mathbf{m}$ to $q$.

## 3.3 Comments

On p. 347 of [8] Thomson deduced that the field momentum of an electric charge plus a small solenoid (magnetic dipole) is due to the electromagnetic fields inside the small solenoid, with value $\mathbf{E}\times\mathbf{m}/c$. He also considered while the momentum in the fields outside the solenoid is zero, it consists of a piece $\mathbf{E}\times\mathbf{m}/c$ associated with the charge, and another piece $-\mathbf{E}\times\mathbf{m}/c$ associated with the magnet. In this view, the sum of the field momenta associated with the magnet is zero, and all the field momentum is associated with the electric charge, as is also

[18] The difference between the magnetic fields of "point" Ampèrian and Gilbertian magnetic dipoles is $4\pi\,\mathbf{m}\,\delta^3(\mathbf{r})$ (see, for example, sec. 5.6 of [49]), which also leads to eq. (13).

[19] A brief version of this argument was given on pp. 348-349 of [8], without relating it to the example of an electric charge plus magnetic dipole which appeared on pp. 347-348.

[20] See, for example, sec. 44 of [52].

suggested by the argument of our sec. 3.2.[21] On p. 348, Thomson noted that if the Ampèrian magnetic dipole were a small permanent magnet (in the field of an electric charge), and this magnet were demagnetized by "tapping", the electric charge should aquire this momentum (although in the view that the field momentum is associated with the magnet, that latter would acquire the momentum).[22]

Thomson did not comment that it this scenario is correct, it would correspond to an "electromagnetic spaceship" that can start from rest, and propel itself in some direction without transfer of momentum to its surroundings.

He did not remark on the ambiguity as to whether the charge or the magnet is set is motion by the disappearance of the field momentum, nor that the resolution of this ambiguity is that the total momentum of the system must be zero (when it is "at rest") [2], such that there exists a "hidden" mechanical momentum in the system equal and opposite to the field momentum. Then, if the field momentum vanishes the "hidden" mechanical momentum does also, and the total momentum of the system remains zero.[23]

That this "hidden" momentum is of order $1/c^2$,[24] and so is a "relativistic" effect, was beyond the scope of discussions in 1904.[25]

# 4 Field Momentum of a Moving Charged Particle

In 1881, Thomson (as a 25-year-old graduate student) noted [56] that the magnetic field energy of a uniform sphere of radius $a$ with electric charge $q$ and velocity $v \ll c$ has the

[21] The field momentum can be computed by a third method, apparently first noted by Furry [10] (see also Appendix B of [3]), $\mathbf{P}_{\rm EM} = \int V^{(C)} \mathbf{J}\, d{\rm Vol}/c^2$, where $V^{(C)}$ is the scalar potential in the Coulomb gauge and $\mathbf{J}$ is the electric current density. This form is not particularly efficient in deducing eq. (1), but it reinforces the impression that the field momentum is associated with the magnetic dipole.

[22] Thomson's discussion was the origin of the Feynman cylinder paradox [59, 60].

[23] For Thomson's particular example, the magnetic moment drops to zero while the static electric field of the charge is unchanged. In this case, the "overt" mechanical momentum of the dipole changes according to $d\mathbf{p}_{m,{\rm overt}}/dt = -\dot{\mathbf{m}} \times \mathbf{E}/c$ (see, for example, sec. IV of [53], the last line on p. 53 of [54]) and sec. 2.5 of [55]), so the final "overt" momentum of the dipole is $\mathbf{p}_{m,{\rm overt}} = \mathbf{m} \times \mathbf{E}/c$ which equals the initial "hidden" mechanical momentum of the magnetic dipole in the electric field [10]. Meanwhile, the falling magnetic moments leads to an induced electric field at the charge $q$, such that the force on the charge is $\mathbf{F} = q\mathbf{E}_{\rm ind} = -q\partial\mathbf{A}/\partial ct = -q\,\dot{\mathbf{m}} \times \mathbf{r}/cr^3 = \dot{\mathbf{m}} \times \mathbf{E}/c = -d\mathbf{p}_{m,{\rm overt}}/dt$. The final ("overt") momentum of the charge is $\mathbf{p}_q = -\mathbf{m} \times \mathbf{E}/c = -\mathbf{p}_{m,{\rm overt}}$, so the final, total momentum of the system is also zero.

[24] As noted after eq. (1), a loop of area $A$ that carries current $I$ has magnetic moment $m = IA/c$, so the field momentum (15) is an effect of order $1/c^2$.

[25] For comments on the character of this "hidden" momentum, see [5].

value,[26,27,28]

$$U_{\rm M} = \frac{1}{2c}\int \mathbf{J}\cdot\mathbf{A}\, d{\rm Vol} = \frac{2q^2}{15a}\frac{v^2}{c^2} = \frac{2U_{\rm E}v^2}{3c^2} \qquad \left(= \frac{q^2}{3a}\frac{v^2}{c^2} \text{ for a spherical shell}\right), \tag{16}$$

where $U_{\rm E} = \int (E^2/8\pi)\, d{\rm Vol}$. Thomson then interpreted the coefficient of $v^2/2$ in the energy $U_{\rm M}$ as mass due to the electromagnetic field,

$$m_{\rm EM} = \frac{4}{3}\frac{U_{\rm E}}{c^2}\,, \tag{17}$$

launching a debate as to how much of particle mass is due to fields that continues to this day.[29,30]

In 1893 (sec. 16 of [31]), Thomson used his expression (4) to compute the field momentum of a uniformly moving charged shell. The derivation is again more compact if we use $\mathbf{B} = \mathbf{v}/c \times \mathbf{E}$,

$$\begin{aligned} \mathbf{P}_{\rm EM} &= \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} = \int \frac{\mathbf{E}\times(\mathbf{v}\times\mathbf{E})}{4\pi c^2}\, d{\rm Vol} = \frac{\mathbf{v}}{c^2}\int \frac{E^2(1-\cos^2\theta)}{4\pi}\, d{\rm Vol} = \frac{4U_{\rm E}}{3c^2}\,\mathbf{v} \\ &= m_{\rm E}\,\mathbf{v}, \end{aligned} \tag{18}$$

with the electromagnetic mass $m_{\rm E}$ as in eq. (17).

# 5 Field Momentum of a Pair of Moving Charged Particles

Thomson was aware that changes in the field momentum should be considered when discussing electromagnetic forces in nonstatic situations (sec. 281 of [9]), and discussed the case of a pair of moving charged particles in pp. 349-354 of [8]. If these charges are regarded as short, isolated current elements, their forces on one another are not equal and opposite,

[26] Thomson's derivation involved setting $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$ (*i.e.*, use of the Coulomb gauge), as favored by Maxwell (sec. 98 of [11] and sec. 617 of [12]). Fitzgerald commented on this procedure in [57], and later came to favor the Lorenz gauge [58] in which the potentials do not have instantaneous components. See also pp. 115-118 of [61], and sec. IIC of [19].

[27] The result (16) is more readily obtained on noting that for $v \ll c$ the electric field $\mathbf{E}$ of the moving charge is the instantaneous static field, while (for any constant speed) the magnetic field is $\mathbf{B} = \mathbf{v}/c \times \mathbf{E}$ (eq. (29) of [30]; see also p. 20 of [7]), such that $U_{\rm M} = \int (B^2/8\pi)\, d{\rm Vol} = (v^2/c^2)\int [E^2(1-\cos^2\theta)/8\pi]\, d{\rm Vol} = 2v^2U_{\rm E}/3c^2$.

[28] The result (16) was verified to hold for any $v < c$ by Heaviside in 1889 [30], which analysis was subsequently noted as implying that the moving sphere is, Fitzgerald-Lorentz contracted [62].

[29] In the author's view, Thomson's 1881 paper [56] marks the beginning of elementary-particle physics (at least in the English-speaking community), a topic avoided by the generations of Ampère and Maxwell (although kept alive in Germany by Weber and his followers, as reviewed, for example, in [63]). An early use of what is now called the Lorentz force law for a charged particle appears in sec. 5 of this paper (although this law was used in Weber's electrodynamics, and appears heavily disguised in sec. 599 of Maxwell's *Treatise* [12]; see also [64]).

[30] The discrepancy between eq. (17) and Einstein's $U = mc^2$ [65] is called the "4/3 problem". Some of the many commentaries on this "perpetual" problem include [66, 67, 68, 69, 70, 71].

which led Ampère [72, 73] to argue that isolated current elements (*i.e.*, free moving charges) cannot exist.

Thomson deduced that the field momentum of the charges $q_1$ and $q_2$, assumed to be in uniform motion with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, is,

$$\mathbf{P}_{\rm EM} = \frac{q_1 q_2}{2c^2 R}[\mathbf{v}_1 + \mathbf{v}_2 + (\mathbf{v}_1 \cdot \hat{\mathbf{R}} + \mathbf{v}_2 \cdot \hat{\mathbf{R}})\,\hat{\mathbf{R}}], \tag{19}$$

where $\mathbf{R}$ points from charge 1 to charge 2. However, Thomson did not explicitly verify that,

$$\frac{d\mathbf{P}_{\rm EM}}{dt} = -\frac{d\mathbf{P}_{\rm mech}}{dt} = -(\mathbf{F}_{12} + \mathbf{F}_{21}), \tag{20}$$

such that $\mathbf{P}_{\rm total} = \mathbf{P}_{\rm mech} + \mathbf{P}_{\rm EM}$ is constant, as was confirmed much later [74].[31]

Thomson did write down (p. 352 of [8]) the energy of the two charged particles as (after some simplification),

$$U = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{q_1 q_2}{R} + \frac{q_1 q_2}{2c^2 R}[\mathbf{v}_1 \cdot \mathbf{v}_2 + (\mathbf{v}_1 \cdot \hat{\mathbf{R}})(\mathbf{v}_2 \cdot \hat{\mathbf{R}})], \tag{21}$$

which is the so-called Darwin Hamiltonian $\mathcal{H}$ [76] except for the absence of quartic corrections associated with "relativistic mass", $m = m_{\rm rest}/\sqrt{1 - v^2/c^2}$.[32]

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[31] For a closely related example, see [75].

[32] Note that $\mathbf{P}_{\rm EM,i} = \partial\mathcal{H}_{\rm EM}/\partial\mathbf{v}_i$ leads to eq. (19) with $\mathbf{P}_{\rm EM,1} = (q_1 q_2/2c^2 R)[\mathbf{v}_2 + (\mathbf{v}_2 \cdot \hat{\mathbf{R}})\,\hat{\mathbf{R}}]$, *etc.*

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