Motion of a Leaky Tank Car

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(December 4, 1989; updated May 31, 2018)

1 Problem

Describe the motion of a tank car initially at rest once an off-center drain opens. The tank car rolls without friction on a horizontal surface,[1] and the water flows out of the drain vertically in the rest frame of the car.

2 Solution

The motion of a leaky tank car is surprisingly complex. We approach a solution in four steps: a brief discussion of the motion, a discussion of the forces that cause the motion, a general analysis, and lastly two detailed examples.

This problem has appeared in recent years on qualifying exams in Russia.[2,3]

2.1 Brief Discussion

There are no external horizontal forces on the system of tank car + water (including the water that has drained out), so horizontal momentum is conserved and the c.m. of the whole system must remain fixed. As the water drains out of the off-center hole, the tank car initially moves opposite to the direction of the drain to keep the c.m. fixed. But, if this motion persisted until all the water drained out, then both the car and the fallen water would have momentum in the same direction. Rather, the car reverses its direction of motion at some time and has a final velocity towards the drain.

2.2 Discussion of the Forces

The water leaves the tank with zero relative horizontal velocity. On recalling the usual rocket problem, there is no propulsion due to the rocket exhaust unless the exhaust has a nonzero velocity relative to the rocket. So in the present problem, there is no rocket action in the usual sense. However, in both problems the forces are due to momentum transfers between the walls of the rocket or tank and the working substance inside. In a rocket the walls absorb momentum from the gas molecules moving outward from the region of combustion; because of the hole at the rear of the rocket this leads to a net forward force.

The case of the leaky tank car is more subtle. The motion begins as a transient response in which the walls of the tank push on the water to move it towards the drain, once the

[1]The wheels of the car must be massless for this condition to hold. Or, consider sliding without friction.

[2]A version of this problem appears in sec. II of [1] without a statement as to the location of the drain. Another version was discussed by the author in [2]. Very recent discussions appear in [3, 4].

[3]Feb. 7, 2022. The problem was posed (and solved) by E.B. Wilson in 1910 [5].

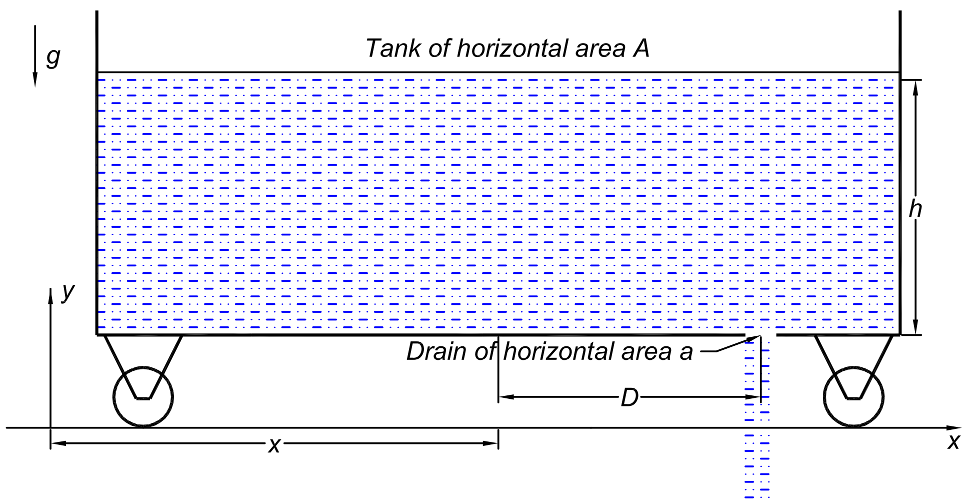
latter is opened. The reaction on the walls of the tank gives it an initial velocity opposite to the direction from the c.m. to the drain.

If the flow rate were constant in time, the tank would roll with constant velocity (see additional discussion in sec. 2.4. But, if the flow rate decreases as a function of time the tank walls (or floor) must slow down the flow towards the drain, which in turn reduces the velocity of the tank. Since the tank + water now has less mass than before, the momentum reabsorbed from the flowing water can actually reverse the initial velocity!

Indeed, the final velocity of the tank car must be opposite to its initial velocity. If not, then the empty tank car + water that has fallen out would all have nonzero momentum in the same direction, in contradiction to the state before the drain was opened.

2.3 Newtonian Analysis, I

We first present a solution based on a very idealized model that avoids detailed description of the internal motion of the water in the tank, even though this is where the "real physics" resides. In Appendix A we give a Newtonian analysis of a somewhat more realistic model that includes horizontal motion of water in the tank associated with the drain, and in Appendix B we give a Lagrangian analysis of this model (whereas the model discussed in this section does not permit a Lagrangian analysis due to its excessive simplicity).



We take $x(t)$ to be the horizontal coordinate of the center of the tank car whose mass (with no water) is m, and suppose that the tank car starts from rest $x = 0$ at time $t = 0$. In our first model, the drain is simply a hole of area a at distance D from the center of the bottom of the tank, and we assume that somehow the water exits the drain with zero horizontal velocity relative to that of the tank car.

The water in the tank has a horizontal surface at height h above the bottom of the tank. The tank has horizontal area $A \gg a$.

We approximate the water as incompressible and inviscid. Then, the equation of continuity for the water (in the rest frame of the tank car) implies that the vertical velocity V of water exiting the drain is related by,

$$\mathbf{V} = -\frac{A}{a}\frac{dh}{dt}\,\hat{\mathbf{y}} = -\frac{A}{a}\dot{h}\,\hat{\mathbf{y}}. \tag{1}$$

2.3.1 Momentum Analysis

The center of mass of the entire system must remain at the origin,

$$0 = [m + M(t)]\, x(t) + \int dM(t')\, X(t, t'), \tag{2}$$

where m is the mass of the tank (without water), $M(t)$ is the mass of the water remaining in the tank, $dM(t')$ is the amount of water that drained out in the interval dt' centered on an earlier time t', and $X(t, t')$ is the horizontal coordinate at time t of the water that drained out at time t'. In writing this we have assumed that the surface of the water in the tank is always horizontal, so that the center of mass of that water is in the center of the tank.

In the interval dt' at an earlier time t', mass $-\dot{M}(t')\, dt'$ of water drains out with horizontal velocity $\dot{x}(t')$ in the lab frame. At time t' the drain was at $x(t') + D$, so at time t the element dM is at $X(t, t') = x(t') + D + (t - t')\, \dot{x}(t')$. Thus the c.m. of the whole system obeys,

$$\begin{aligned} 0 &= [m + M(t)]\, x(t) + \int_0^t dt'\, [-\dot{M}(t')]\, [x(t') + D + (t - t')\, \dot{x}(t')] \\ &= [m + M(t)]\, x(t) - t \int_0^t dt'\, \dot{M}(t')\, \dot{x}(t') - \int_0^t dt'\, \dot{M}(t')\, [x(t') + D - t'\, \dot{x}(t')]. \end{aligned} \tag{3}$$

While an integral equation is not the usual starting point in a mechanics problem, it has the advantage here of integrating over the unknown transient forces that occur when the drain is first opened.

We take time derivatives of eq. (3) to find the equation of motion. The first derivative yields,

$$0 = (m + M)\, \dot{x} - \int_0^t dt'\, \dot{M}(t')\, \dot{x}(t') - \dot{M} D. \tag{4}$$

We recognize eq. (4) as stating that the total momentum of the system is always zero: the first term is the momentum in the tank + water supposing there is no relative motion of the tank and water, the second term is the momentum of the water that has left the tank, and (hence) the third term is the momentum of the water in the tank as measured in the rest frame of the tank.

We can use eq. (4) to examine the "initial" condition on $\dot{x}$ at an arbitrarily small but positive time at the drain is first opened,

$$\dot{x}(0^+) = \frac{\dot{M}(0^+) D}{m + M(0^+)} < 0. \tag{5}$$

Whatever the form of flow rate $\dot{M}(t)$ that could be arranged, $\dot{M}(0)$ is negative so long as the water is draining out, and $\dot{x}(0^+)$ is in the opposite direction from the c.m. of the tank car to the drain (*i.e.*, in the $-x$ direction).

Since eq. (5) is the limit of (4) as $t \to 0$ from positive values, it contains the result of the transient at $t = 0$. So, while the velocity of the tank was zero before the drain is opened, it has a finite value just after the drain is opened and the flow of water inside the tank has been established. In reaction to the forces from the tank on the water, there is an impulse from the water on the tank that creates the initial velocity (5).

On taking the derivative of eq. (4), we find,

$$0 = (m + M)\,\ddot{x} - \ddot{M}D \qquad \Big(F = (M + m)\ddot{x} = \ddot{M}D\Big)\,. \tag{6}$$

This can be interpreted as indicating that the force on the tank + water therein is just the reaction force $\ddot{M}D$ of the acceleration of the water relative to the tank. Because the water leaves the tank with zero relative velocity, the momentum $(m+M)\,\dot{x}$ is reduced without any reaction force; hence the simplification of (6) compared to (4).

While $\dot{M}$ is always less than zero, $\ddot{M}$ is positive for any realistic flow out of a constant-sized drain hole, and there will be a force in the $+x$ direction. This force arises as the walls of the tank arrest the motion of the water towards the drain, so the water can leave the tank with zero relative velocity. To guarantee the latter condition, the drain might have to be placed in a sump whose vertical walls can assist in absorbing the horizontal momentum of the water flow (as considered in the Appendices).

The interpretations of eqs. (4) and (6) given above have taken the tank + water therein as the subsystem of interest. The reader might prefer to give emphasis to the tank alone. Then, eq. (4) should be rewritten as,

$$m\dot{x} = -\left[(M\dot{x} - \dot{M}D) - \int_0^t dt'\,\dot{M}(t')\,\dot{x}(t')\right], \tag{7}$$

where the term in square bracket is the momentum of the water inside the tank relative to the lab frame. Taking the time derivative,

$$m\ddot{x} = -M\ddot{x} - \dot{M}\dot{x} + \ddot{M}D + \dot{M}\dot{x}. \tag{8}$$

There are four forces on the tank: $-M\ddot{x}$ is the horizontal inertial force of the water in the tank back on the walls of the tank that are accelerating the water; $-\dot{M}\dot{x}$ is a correction to the inertial force of the water because the amount of water inside the tank is changing; $\ddot{M}D$ is the force from the water onto the tank near the drain hole where the horizontal velocity of the water is slowed down until it leaves with zero relative velocity; $\dot{M}\dot{x}$ is the reaction force of the water that leaves the tank back on the tank. As previously noted, because the water leaves the tank with zero relative horizontal velocity, the reaction force of the water leaving the tank exactly cancels the correction to the inertial force of the water left in the tank.

On writing eq. (6) as,

$$\ddot{x} = \frac{\ddot{M}D}{m + M}\,, \tag{9}$$

we can integrate this to find,

$$\dot{x} = \dot{x}(0) + D\int_0^t dt'\,\frac{\ddot{M}(t')}{m + M(t')} = D\left[\frac{\dot{M}(t)}{m + M(t)} + \int_0^t dt'\left(\frac{\dot{M}(t')}{m + M(t')}\right)^2\right], \tag{10}$$

where we have integrated by parts to obtain the second form. When all the water runs out, $\dot{M} \to 0$, but the integral is positive definite, so the final velocity must be positive. For any finite mass m of the car, it will return to the origin after some time, and continue moving in the $+x$ direction.

2.3.2 Energy Analysis

If energy is conserved in the water flow, and if (in the rest frame of the tank) we ignore the kinetic energy of the water in the tank, then the water drains out with a vertical velocity V given by,

$$V^2 = 2gh, \tag{11}$$

where h is the height of the water remaining in the tank car. The mass $M(t) = \rho A h(t)$ of water in the tank then varies according to,

$$\dot{M} = \rho a \dot{h} = -\rho a V = -\rho a \sqrt{2gh} = -a\sqrt{\frac{2g\rho M}{A}} \equiv -2\sqrt{M}S\,, \tag{12}$$

where A and a are the horizontal areas of the tank and drain, respectively, and $S = a\sqrt{g\rho/2A}$. This can be integrated to give,

$$M(t) = \left(\sqrt{M_0} - St\right)^2, \tag{13}$$

where M_0 is the mass of the water when the drain is first opened. All the water has drained from the tank at time $t = \sqrt{M_0}/S$.

From eq. (5), the initial velocity is negative,

$$\dot{x}(0) = \frac{D\dot{M}(0)}{m + M(0)} = -\frac{2DS\sqrt{M_0}}{m + M_0} < 0. \tag{14}$$

From eqs. (12) and (13), we have that $\ddot{M} = 2S^2$, so from eq. (9) the acceleration of the tank car is, for $0, < t < \sqrt{M_0}/S$,

$$\ddot{x} = \frac{D\,\ddot{M}}{m + M} = \frac{2DS^2}{m + (\sqrt{M_0} - St)^2}, \tag{15}$$

which is always positive. We integrate this to find,

$$\dot{x} = 2DS\left[-\frac{\sqrt{M_0}}{m + M_0} + \frac{1}{\sqrt{m}}\left(\tan^{-1}\sqrt{\frac{M_0}{m}} - \tan^{-1}\frac{\sqrt{M_0} - St}{\sqrt{m}}\right)\right], \tag{16}$$

using $\dot{x}(0)$ from eq. (14). When the tank is empty at time $t = \sqrt{M_0}/S$, this becomes,

$$\dot{x}_{\rm empty} = 2DS\left[-\frac{\sqrt{M_0}}{m + M_0} + \frac{1}{\sqrt{m}}\tan^{-1}\sqrt{\frac{M_0}{m}}\right]. \tag{17}$$

This is positive for any combination of finite masses $m \le 0$ and $M_0 > 0$, but for large tank masses m, $\dot{x}_{\rm empty} \to (4/3)DSM_0^{3/2}/m^2$, which approaches zero. Hence, the tank car has reversed its direction at some time $t_{\rm reverse} < \sqrt{M_0}/S$, *i.e.*, when $\dot{x}(t_{\rm reverse}) = 0$,

$$t_{\rm reverse} = \frac{\sqrt{M_0}}{S} - \frac{\sqrt{m}}{S}\tan\left(\tan^{-1}\sqrt{\frac{M_0}{m}} - \frac{\sqrt{mM_0}}{m + M_0}\right). \tag{18}$$

We integrate $\dot{x}$, eq. (16), to find,

$$\begin{aligned}\frac{x}{D} &= -2\frac{\sqrt{M_0}St}{m+M_0} + \ln\frac{m+M_0}{m+(\sqrt{M_0}-St)^2} \\ &\quad -2\frac{\sqrt{M_0}-St}{\sqrt{m}}\left(\tan^{-1}\sqrt{\frac{M_0}{m}} - \tan^{-1}\frac{\sqrt{M_0}-St}{\sqrt{m}}\right). \qquad (19)\end{aligned}$$

All three terms vanish at $t = 0$; the first term is just $\dot{x}(0)t$. When the tank is empty at time $t = \sqrt{M_0}/S$, its position is,

$$\frac{x_{\rm empty}}{D} = -2\frac{M_0}{m+M_0} + \ln\frac{m+M_0}{m}. \qquad (20)$$

The critical value of the mass of the tank car that just returns to the origin when empty is $m = 0.255M_0$. For larger masses m, the tank car is at negative x when the tank goes empty. Since $\dot{x}_{\rm empty}$ is always positive, the tank car passes through the origin at some later time.

2.3.3 Massless Tank Car

We also consider the limit of a massless tank car. Then, from eq. (9),

$$\begin{aligned}\ddot{x} &= \frac{2DS^2}{(\sqrt{M_0}-St)^2}, \qquad (21) \\ \dot{x} &= -\frac{4DS}{\sqrt{M_0}} + \frac{2DS}{\sqrt{M_0}-St}, \qquad (22)\end{aligned}$$

using $\dot{x}(0) = -2DS/\sqrt{M_0}$, and,

$$\frac{x}{D} = -\frac{4St}{\sqrt{M_0}} + 2\ln\frac{\sqrt{M_0}}{\sqrt{M_0}-St}. \qquad (23)$$

Here the tank car reverses its direction at time $t = \sqrt{M_0}/2S$ (*i.e.*, when it is still 3/4 full) and at position $x_{\rm min} = -0.61D$. Then, at some later time it passes the origin and moves toward large x as it empties.

2.3.4 The Water Drains at a Constant Rate

It is instructive to contrast the above results with the case that the water drains out at a constant rate, as might be arranged by a pump at the drain,

$$M(t) = M_0 - Rt, \qquad \dot{M} = -R, \qquad \text{and} \qquad \ddot{M} = 0. \qquad (24)$$

Because the flow rate is uniform no (horizontal) momentum is transferred from the flowing water to the tank walls once the flow has been established. Hence, there are no further (horizontal) forces between the water and the tank, and the velocity of the tank is constant at its initial value. Then,

$$\dot{x}(t) = \dot{x}(0) = -\frac{DR}{m+M_0}, \qquad (25)$$

holds until the tank goes empty at time $t = M_0/R$. Its position is then $x_{\rm empty}/D = -M_0/(m + M_0) > -1$.

However, we seem to have a paradox: at time $t = M_0/R$ the water has all emptied out of the tank and now has momentum $M_0\dot{x}(0)$, and the tank appears to have momentum $m\dot{x}(0)$, both of which are negative! The water actually does have the momentum as stated, and we infer that since the total momentum must be zero, the tank takes on a final velocity,

$$\dot{x}_{\rm final} = -\frac{M_0\dot{x}(0)}{m}. \tag{26}$$

This arises due to another transient force at time $t = M_0/R$. The momentum of the water in the tank relative to the tank has remained constant at $-\dot{M}D$ even though the mass of water that contains this momentum approaches zero. At time $t = M_0/R$, all this momentum must suddenly be transferred to the tank, which leads to an impulse and, hence, to the final velocity found above. In practice, viscous drag will prevent the water from acquiring the infinite velocity implied above, so the leak rate will decrease and the momentum in the water will be transferred to the tank over a short interval just before the water runs out.

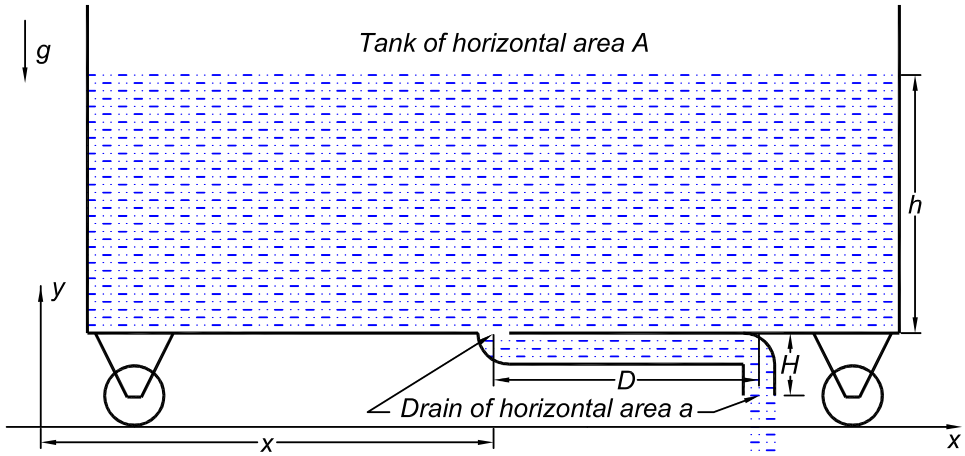
In the present example, it is also easy to keep track of the momentum of the water in the tank relative to the lab frame. The impulse at $t = 0$ gives the tank the velocity $\dot{x}(0)$ and, hence, momentum $m\dot{x}(0)$. The initial momentum of the water in the tank is consequently $-m\dot{x}(0)$. By time t, water of total mass Rt has left the tank taking momentum $Rt\dot{x}(0)$ with it. Since the tank's momentum is unchanged, the momentum of the water in the tank is $-(m + Rt)\dot{x}(0)$. Just before the tank goes empty, the water in it contains momentum $-(m + M_0)\dot{x}(0) > 0$, which must be transferred to the tank in the final impulse. Again, we arrive at the value of $\dot{x}_{\rm final}$ found above.

2.3.5 Practicalities

As a final remark, we consider whether the motion of a leaky tank car can be observed in practice. For a railroad tank car that is 20 m long with a 5×5 m^2 cross section and has a 10×10 cm^2 drain at one end, the initial velocity would be about 6 cm/sec according to eq. (14). The forces that produce this velocity are likely too small to overcome friction, and no motion would be observed. Instead, one might use an air track from a physics teaching lab, and build a sliding tank of mass, say, 2 grams that could hold 10 grams of water. The drain hole could then be 1 cm off center. If the hole has area of 1 mm^2, the initial velocity would be about 0.05 cm/sec. This is rather small, but should be observable. Care must be given that external forces during the opening of the hole do not impart a comparable velocity.

A Appendix: Newtonian Analysis, II

In this Appendix we consider a model (suggested by J. Otto) of the horizontal motion, relative to the tank, of water inside it, as sketched below. For a related analysis, see [2].



We again take $x(t)$ to be the horizontal coordinate of the center of the tank car whose mass (with no water) is m, and suppose that the tank car starts from rest $x = 0$ at time $t = 0$. The drain involves a hole of area a in the center of the bottom of the tank, connected to a horizontal pipe of area a and length D, ending in a vertical section of area a and height H that causes the water to exit the drain with vertical velocity V in the rest frame of the tank. The tank has horizontal area $A \gg a$.

The water in the tank has a horizontal surface at height h above the bottom of the tank.

We approximate the water as incompressible and inviscid. The equation of continuity for the water then tell us that,

$$\mathbf{V} = -\frac{A}{a}\dot{h}\,\hat{\mathbf{y}}. \tag{27}$$

The system of tank plus water therein has two degrees of freedom, x and h, so we seek two equations of motion, via conservation of horizontal momentum and conservation of energy (ignoring rolling friction and the viscosity of water).

A.1 Momentum Analysis

The mass M of the water in the tank is,

$$M = \rho[Ah + a(D + H)], \tag{28}$$

where ρ is the mass density of water. The total mass of the tank plus water therein is,

$$m_{\text{in}} = m + M = m + \rho[Ah + a(D + H)], \tag{29}$$

whose time rate of change is, so long as $h \geq 0$,

$$\dot{m}_{\text{in}} = \dot{M} = \rho A\dot{h}. \tag{30}$$

The horizontal momentum of the tank + water therein at time t is,

$$p_{\text{in},x} = [m + \rho(Ah + aH)]\dot{x} + \rho a D\left(\dot{x} - \frac{A}{a}\dot{h}\right) = \{m + \rho[Ah + a(D+H)]\}\dot{x} - \rho AD\dot{h}, \quad (31)$$

since the water in the horizontal section of the drainpipe has mass ρaD and horizontal velocity $\dot{x} - (A/a)\dot{h}$. The horizontal momentum of the tank + water still therein at time $t + dt$ has changed by amount,

$$dp_{\text{in},x} = \{m + \rho[Ah + a(D+H)]\}\ddot{x}\,dt + \rho A\dot{h}\,dt\,\dot{x} - \rho AD\ddot{h}\,dt, \quad (32)$$

to order dt, while the horizontal momentum of the water newly outside the pipe is,

$$dp_{\text{out},x} = -\rho A\dot{h}\,dt\,\dot{x}, \quad (33)$$

since the water that exits the pipe has horizontal velocity $\dot{x}$ in the lab frame, and the volume of newly exited water is $-A\dot{h}\,dt$.

If the pipe rolls without friction, horizontal momentum is conserved, $dp_{\text{in},x} + dp_{\text{out},x} = 0$, such that,[4]

$$\{m + \rho[Ah + a(D+H)]\}\ddot{x} = (m+M)\ddot{x} = \rho AD\ddot{h} = \ddot{M}D, \quad (35)$$

recalling eqs. (28) and (30). This is the same as eq. (9) of sec. 2.3.1 above.

In an $F = ma$ interpretation, the horizontal acceleration $\ddot{x}$ of the tank + water therein is due to the reaction force associated with the horizontal acceleration $\ddot{h}$ relative to the tank of the mass ρaD of water in the horizontal segment of the drainpipe.

Alternatively, in a view that emphasizes the tank of mass m, we could write eq. (35) as,

$$m\ddot{x} = -M\ddot{x} - \dot{M}\dot{x} + \rho aD\ddot{h} + \dot{M}\dot{x}, \quad (36)$$

where $-M\ddot{x}$ is the inertial force of the water in the pipe back on the walls of the pipe that are accelerating the water; $-\dot{M}\dot{x}$ is a correction to the inertial force of the water because the amount of water inside the pipe is changing; $\rho aD\ddot{h}$ is the force of the water on the right bend of the drainpipe where the horizontal velocity water relative to the pipe is reduced from $\dot{h}$ to zero; and $\dot{M}\dot{x}$ is the reaction force of the water that leaves the pipe back on the pipe. Because the water leaves the pipe with zero relative horizontal velocity, the reaction force of the water leaving the pipe exactly cancels the correction to the inertial force of the water left in the pipe.

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[4] Alternatively, we could analyze the horizontal position x_{cm} of the system which must be fixed if the pipe rolls/slides without friction.

We take the center of the tank to start from rest at $x = 0$ at time $t = 0$, such that at time t,

$$m_{\text{total}}x_{\text{cm}} = mx + \rho Ahx + \rho aD(x + D/2) + \rho aH(x + D) + \int_0^t (-\rho A\dot{h}\,dt')X(t,t'), \quad (34)$$

where $X(t,t') = x(t') + \dot{x}(t')(t - t')$ is the present position of the element of water of mass $-\rho a\dot{h}(t')\,dt'$ that left the drain at time t' with zero relative horizontal velocity, and hence with lab-frame horizontal velocity $\dot{x}(t')$. Taking the time derivative of eq. (34) we find eq. (37), and taking the time derivative of eq. (37) we find eq. (35).

A formal time integration of eq. (35) yields (using an integration by parts), for the system starting from rest at $t = 0$,

$$m_{\rm in}\dot{x} - \int_0^t \dot{m}_{\rm in}(t')\dot{x}(t')\,dt' = \rho a D\dot{h}, \tag{37}$$

$$\dot{x} = \frac{\rho a D}{m_{\rm in}}\dot{h} + \frac{\rho a}{m_{\rm in}}\int_0^t \dot{h}(t')\dot{x}(t')\,dt', \tag{38}$$

recalling that $\dot{m}_{\rm in} = \rho a\dot{h}$. At time $t = 0^+$, when the water starts to drain out of the tank, the integral in eq. (37) is negligible and we have that,

$$\dot{x}(0^+) = \frac{\rho a D}{m_{\rm in}}\dot{h}(0^+) = \frac{D\dot{M}(0)}{m + M(0)} < 0, \tag{39}$$

as previously found in eq. (14). That is, the tank initially moves to the left, as the water drains out of the right end of the drainpipe, such that the center of mass of the system stays at rest.

As time increases, the first term on the right of eq. (38) remains negative, while the second term becomes increasingly positive, such that the sign of $\dot{x}$ can reverse. Indeed, we noted in sec. 2.1 that the sign of $\dot{x}$ must eventually reverse, as the momentum of the system would eventually be nonzero and negative if $\dot{x}$ were always negative. However, it can still be that the height h of the water in the tank goes to zero before $\dot{x}$ changes sign.

A.2 Energy Analysis

The kinetic energy of the tank car + water therein at time t in the lab frame is,

$$T_{\rm in} = [m + \rho(Ah + aH)]\frac{\dot{x}^2}{2} + \rho Ah\frac{\dot{h}^2}{2} + \rho aH\frac{A^2}{a^2}\frac{\dot{h}^2}{2} + \rho aD\frac{(\dot{x} - A\dot{h}/a)^2}{2}\,. \tag{40}$$

The kinetic energy of the tank car + water still therein at time $t + dt$ has changed by amount,

$$\begin{aligned} dT_{\rm in} \;&=\; [m + \rho(Ah + aH)]\,\dot{x}\,\ddot{x}\,dt + \rho A(h + AH/a)\dot{h}\ddot{h}\,dt + \rho A\dot{h}\,dt\frac{(\dot{x}^2 + \dot{h}^2)}{2} \\ &\quad +\rho aD(\dot{x} - A\dot{h}/a)(\ddot{x} - A\ddot{h}/a)dt, \end{aligned} \tag{41}$$

to order dt, while the kinetic energy of the water newly outside the pipe is,

$$dT_{\rm out} = -\rho A\dot{h}\,dt\frac{(\dot{x}^2 + A^2\dot{h}^2/a^2)}{2}\,. \tag{42}$$

Assuming that no dissipative forces are present, the change in kinetic energy equals to work W done on the system by gravity as mass element $-\rho A\dot{h}$ is transferred from height h to height $-H$,

$$dT_{\rm in} + dT_{\rm out} = W = -\rho A\dot{h}g(h + H) = d\text{Potential Energy}. \tag{43}$$

That is,

$$\{m + \rho[Ah + a(D+H)]\}\,\dot x\,\ddot x + \rho A[h + A(D+H)/a]\dot h\ddot h - \rho AD(\dot x\ddot h + \dot h\ddot x)$$
$$-\rho A\frac{\dot h^2}{2}\left(\frac{A^2}{a^2} - 1\right) + \rho A(h+H)g\dot h = 0. \tag{44}$$

The first term can be eliminated using the momentum equation (35), leading to[5,6]

$$D\ddot x = \left[h + \frac{A}{a}(D+H)\right]\ddot h - \frac{\dot h^2}{2}\left(\frac{A^2}{a^2} - 1\right) + (h+H)g. \tag{50}$$

$$\ddot x = \frac{h + A(D+H)/a}{D}\ddot h - \frac{\dot h^2}{2D}\left(\frac{A^2}{a^2} - 1\right) + \frac{h+H}{D}g \tag{51}$$

A.3 Comments

The model of the leaky tank car considered in this Appendix is somewhat more realistic than that of sec. 2.3 above, and offers that satisfaction that the equations of motion can be deduced by a Lagrangian method (Appendix B) as well as by a Newtonian method. The equation of motion (35) for coordinate x can be interpreted as being the same as eq. (9)

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[5] If $D = 0$ and $a = A$, the system is simply a vertical pipe with vertically falling water, and eq. (50) reduces to $\ddot h = -g$ as expected.

[6] We could also work in the accelerated frame of the pipe, if we take into account the apparent horizontal "coordinate" force $-\mathsf{m}\ddot x$ on any mass m in the accelerated frame. The kinetic energy of the water in the tank at time t in the accelerated frame is,

$$T^\star_{\rm in} = \rho Ah\frac{\dot h^2}{2} + \rho a(D+H)\frac{A^2}{a^2}\frac{\dot h^2}{2}\,. \tag{45}$$

The kinetic energy of the water still inside the tank at time $t + dt$ has changed by amount,

$$dT^\star_{\rm in} = \rho Ah\dot h\ddot h\,dt + \rho A\dot h\,dt\frac{\dot h^2}{2} + a(D+H)]\frac{A^2}{a^2}\dot h\ddot h\,dt, \tag{46}$$

to order dt, while the kinetic energy of the water newly outside the pipe is,

$$dT^\star_{\rm out} = -\rho A\dot h\,dt\frac{A^2}{a^2}\frac{\dot h^2}{2}\,. \tag{47}$$

Assuming that no dissipative forces are present, the change in kinetic energy equals the work W done by gravity as mass element $-\rho A\dot h$ is transferred from height h to height $-H$, plus the work done by the coordinate force $-\mathsf{m}\ddot x$ on the mass of water $\mathsf{m} = \rho aD$ that moves horizontal distance $-(A/a)\dot h dt$,

$$dT^\star_{\rm in} + dT^\star_{\rm out} = W^\star = -\rho A\dot h g(h+H) + \rho AD\dot h\,dt\,\ddot x. \tag{48}$$

That is,

$$\rho A\left[h + \frac{A}{a}(D+H)\right]\dot h\ddot h - \rho A\dot h\frac{\dot h^2}{2}\left(\frac{A^2}{a^2} - 1\right) - \rho DA\dot h\ddot x + \rho(h+H)Ag\dot h = 0, \tag{49}$$

which becomes eq. (50) after dividing by $\rho A\dot h$. This equation can be regarded as an example of the so-called extended Bernoulli equation, eq. (12) of [7], for nonsteady fluid flow in possibly accelerating frames.

deduced from the model of sec. 2.3, while the equation of motion (50) for coordinate h is more subtle than the approximation in sec. 2.3.2 that all the potential energy released by the flow of water out of the tank (efflux) appears as the kinetic energy of the efflux in the rest frame of tha tank car. As such, the motion described by eqs. (16) and (19) slightly overestimated the (tiny) velocity of the tank car and the consequent changes in its x-coordinate. However, the improved equations of motion (35) and (50) don't readily lead to improved insights as to the "paradoxical" motion of the leaky tank car that were obtained in sec. 2.3.

B Appendix: A Lagrangian Analysis

A Lagrangian approach to variable-mass problems has been given in [8, 9], which was applied in Appendix B of [10] to a leaky tank at rest (Torricelli's problem), and in sec. 2.2 of [11] to a leaky bucket suspended from a spring. Lagrangian analyses of two variants of leaky tank cars have been given in [3].

This method considers the kinetic energy $T(q_k, \dot{q}_k, t)$ (but not the potential energy) of a system described by coordinates q_k, and supplements the generalized forces of Lagrange with additional terms, related to a so-called *control volume* whose velocity is $\mathbf{w}$, according to eq. (5.6) of [8] and eq. (1) of [9],

$$\frac{d}{dt}\frac{\partial T_w}{\partial \dot{q}_k} - \frac{\partial T_w}{\partial q_k} + \int \frac{\partial \tilde{T}}{\partial \dot{q}_k}(\mathbf{v} - \mathbf{w}) \cdot d\mathbf{Area} - \int \tilde{T}\frac{\partial(\mathbf{v} - \mathbf{w})}{\partial \dot{q}_k} \cdot d\mathbf{Area} = Q_k, \qquad (52)$$

where T_w is the kinetic energy within the control volume, $\tilde{T}$ is the kinetic energy per unit volume, $\mathbf{v}$ is the velocity of the material at a point in the system, and the generalized forces Q_k are related to the external forces on the system by,

$$Q_k = \sum_i \mathbf{F}_i^{\rm ext} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}\,, \qquad (53)$$

supposing the system to consist of particles with mass m_i at positions $\mathbf{r}_i$ subject to external forces $\mathbf{F}_i^{\rm ext}$.

A difficulty with this approach the to present example, in the model sketched on p. 2, is that the formalism of eqs. (52)-(53) does not involve the distance D of the drain from the center of the tank, and hence cannot be used to deduce the equation of motion (8). However, if we consider the model shown on p. 8, the Lagrangian method can be applied.

In the present example we take the system (*i.e.*, the control volume) to be the tank car (of mass m) and the water (of density ρ) therein, which system can be characterized by two coordinates, x = horizontal position of the center of the tank, and h = height of water in the tank (whose horizontal cross section has area A independent of height). The area drain consist of an outlet at the bottom center of the tank, which is connected by a horizontal pipe of small area $a \ll A$ and length D to a vertical section of pipe of area a and height H, is small compared to A. We assume that this configuration results in a flow of water whose velocity V at the exit of the drain is purely vertical in the rest frame of the tank car.

We take the surface of the control volume to be just outside the physical surface of the tank. The velocity of the control volume is then $\mathbf{w} = \dot{\mathbf{x}}$. There is no matter of the system on the surface of the control volume, except for the water that is exiting the drain with vertical velocity $\mathbf{V} = -V\,\hat{\mathbf{y}}$, which is related to $\dot{h}$ by the continuity equation (approximating the water as incompressible),

$$V = -\frac{A}{a}\dot{h}. \tag{54}$$

The horizontal velocity of the exiting water is, of course $\dot{\mathbf{x}} = \mathbf{w}$, so at the exit, $\mathbf{v} - \mathbf{w} = \dot{\mathbf{x}} + \mathbf{V} - \mathbf{w} = \mathbf{V} = A\dot{h}\,\hat{\mathbf{y}}/a$. The kinetic energy within the control volume is given by eq. (40),

$$T_w = T_{\rm in} = [m + \rho(Ah + aH)]\frac{\dot{x}^2}{2} + \rho Ah\frac{\dot{h}^2}{2} + \rho aH\frac{A^2}{a^2}\frac{\dot{h}^2}{2} + \rho aD\frac{(\dot{x} - A\dot{h}/a)^2}{2}. \tag{55}$$

The kinetic energy per unit volume at the drain is,

$$\tilde{T}_{\rm drain} = \frac{\rho(\dot{x}^2 + V^2)}{2} = \frac{\rho}{2}\left(\dot{x}^2 + \frac{A^2}{a^2}\dot{h}^2\right). \tag{56}$$

The area vector at the drain is $d\mathbf{Area} = -a\,\hat{\mathbf{y}}$.

The external force on the system is $-(m + M)g\,\hat{\mathbf{y}} = -\{m + \rho[Ah + a(D + H)]\}g\,\hat{\mathbf{y}}$.

B.1 Equation of Motion for Coordinate x

The generalized force Q_x is,[7]

$$Q_x = -\sum_i \{m + \rho[Ah + a(D + H)]\}_i\, g\,\hat{\mathbf{y}} \cdot \frac{\partial \mathbf{r}_i}{\partial x} = -\sum_i \{m + \rho[Ah + a(D + H)]\}_i\, g\,\hat{\mathbf{y}} \cdot \hat{\mathbf{x}} = 0. \tag{57}$$

From the kinetic energy (55) we have,

$$\frac{d}{dt}\frac{\partial T_w}{\partial \dot{x}} - \frac{\partial T_w}{\partial x} = \{m + \rho[Ah + a(D + H)]\}\ddot{x} + \rho A\dot{h}\dot{x} - \rho AD\ddot{h}. \tag{58}$$

From eq (56) we have,

$$\frac{\partial \tilde{T}_{\rm drain}}{\partial \dot{x}} = \rho\dot{x}. \tag{59}$$

Then,

$$\int_{\rm drain} \frac{\partial \tilde{T}_{\rm drain}}{\partial \dot{x}}(\mathbf{v} - \mathbf{w}) \cdot d\mathbf{Area} = \rho\dot{x}\left(\frac{A}{a}\dot{h}\,\hat{\mathbf{y}}\right) \cdot (-a\,\hat{\mathbf{y}}) = -\rho A\dot{h}\dot{x}, \tag{60}$$

and since $\partial(\mathbf{v} - \mathbf{w})/\partial\dot{x} = \partial\mathbf{V}/\partial\dot{x} = 0$, we have that

$$\int_{\rm drain} \tilde{T}_{\rm drain}\frac{\partial(\mathbf{v} - \mathbf{w})}{\partial \dot{x}} \cdot d\mathbf{Area} = 0. \tag{61}$$

Altogether, the equation of motion for coordinate x, according to eq. (52), is,

$$\{m + \rho[Ah + a(D + H)]\}\ddot{x} + \rho A\dot{h}\dot{x} - \rho AD\ddot{h} - \rho A\dot{h}\dot{x} = 0, \tag{62}$$

$$(m + M)\ddot{x} = \rho AD\ddot{h} = D\ddot{M}, \tag{63}$$

recalling eqs. (28) and (30). This is the same as eqs. (9) and (35) above.

[7]We recall that Lagrange's method distinguishes between external and constraint forces. In the present example, the upward normal force of the rails on tank car is a constraint force, and so is not included in the computation of the generalized force.

B.2 Equation of Motion for Coordinate h

The generalized force Q_h is (delicately),

$$
\begin{aligned}
Q_h &= -\sum_i \{m + \rho[Ah + a(D+H)]\}_i \, g \, \hat{\mathbf{y}} \cdot \frac{\partial \mathbf{r}_i}{\partial h} \\
&= -\sum_i \rho A h_i \, g \, \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} - \sum_i \rho a H_i \, g \, \hat{\mathbf{y}} \cdot \frac{A}{a} \hat{\mathbf{y}} = -\rho A (h + H) \, g, \qquad (64)
\end{aligned}
$$

in that the y-coordinates of particles in the tank, and in the horizontal pipe of the drain, are not related to the water level h, while particles in the vertical section of the drainpipe move by $A\delta h/a$ when particles in the tank move by δh. From the kinetic energy (55) we have,

$$
\frac{d}{dt}\frac{\partial T_w}{\partial \dot{h}} - \frac{\partial T_w}{\partial h} = \rho A \left[h + \frac{A}{a}(D+H)\right] \ddot{h} - \rho A D \ddot{x} + \rho A \dot{h}^2 - \rho A \frac{\dot{x}^2 + \dot{h}^2}{2}. \qquad (65)
$$

From eq. (56) we have,

$$
\frac{\partial \tilde{T}_{\rm drain}}{\partial \dot{h}} = \rho \frac{A^2}{a^2} \dot{h}. \qquad (66)
$$

Then,

$$
\int_{\rm drain} \frac{\partial \tilde{T}_{\rm drain}}{\partial \dot{h}} (\mathbf{v} - \mathbf{w}) \cdot d\mathbf{Area} = \rho \frac{A^2}{a^2} \dot{h} \left(\frac{A}{a} \dot{h} \, \hat{\mathbf{y}} \right) \cdot (-a \, \hat{\mathbf{y}}) = -\rho \frac{A^3}{a^2} \dot{h}^2, \qquad (67)
$$

and since $\partial \mathbf{V}/\partial \dot{h} = A \, \hat{\mathbf{y}}/a$, we have that,

$$
\int_{\rm drain} \tilde{T}_{\rm drain} \frac{\partial (\mathbf{v} - \mathbf{w})}{\partial \dot{h}} \cdot d\mathbf{Area} = \frac{\rho}{2} \left(\dot{x}^2 + \frac{A^2}{a^2} \dot{h}^2 \right) \left(\frac{A}{a} \, \hat{\mathbf{y}} \right) \cdot (-a \, \hat{\mathbf{y}}) = -\frac{\rho A}{2} \left(\dot{x}^2 + \frac{A^2}{a^2} \dot{h}^2 \right). \qquad (68)
$$

Altogether, the equation of motion for coordinate h according to eq. (52) is,

$$
\begin{aligned}
&\rho A \left[h + \frac{A}{a}(D+H)\right] \ddot{h} - \rho A D \ddot{x} + \rho A \dot{h}^2 - \rho A \frac{\dot{x}^2 + \dot{h}^2}{2} - \rho \frac{A^3}{a^2} \dot{h}^2 + \frac{\rho A}{2} \left(\dot{x}^2 + \frac{A^2}{a^2} \dot{h}^2 \right) \\
&= -\rho A (h + H) \, g, \qquad (69)
\end{aligned}
$$

$$
D \ddot{x} = \left[h + \frac{A}{a}(D+H)\right] \ddot{h} - \frac{\dot{h}^2}{2} \left(\frac{A^2}{a^2} - 1 \right) + (h + H) \, g, \qquad (70)
$$

as previously found in eq. (50).

Acknowledgment

The author wishes to thank F.D.M. Haldane for introducing him to this problem, and J.G. Heinrich and J. Otto for useful discussions.

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