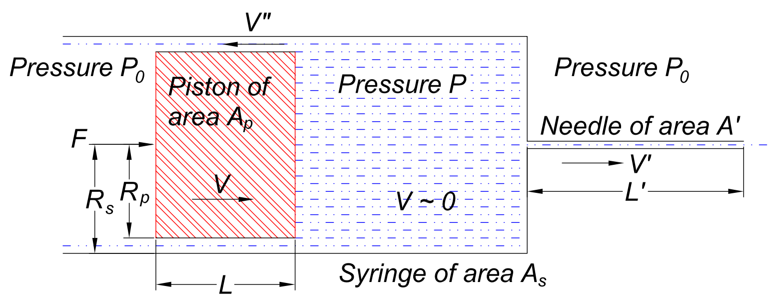
Leaky Syringe

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(August 10, 2014; updated March 17, 2025)

1 Problem

Discuss the fluid flow in a syringe whose piston has area $A_p = \pi R_p^2$ smaller than the area $A_s = \pi R_s^2$ of the cylindrical body, when the piston is force to move at constant velocity V.



This problem was suggested by Johann Otto.

2 Solution

When the area of a fluid flow changes abruptly, the fluid flow may pull away from the wall in the region of smaller area, which effect is called the *vena contracta*, as first noted by Torricelli in 1643 [1].

In the present problem, this could mean that the fluid flow inside the needle (where the average velocity is V') and in the annulus between the piston and the outer cylinder (where the average velocity is V'') has areas smaller than that of the needle of radius R' and the annulus between radii R_s of the outer cylinder and R_p of the piston. If so, there would be five unknown quantities, the pressure P in the syringe (where the flow velocity is approximately zero), the flow velocities V' and V'', and the two reduced areas associated with those flow velocities.

However, there would then be only three equations to determine these five unknowns: mass conservation (assuming an incompressible fluid), and energy and momentum conservation (in the sense that the energy and momentum gained by the fluid is supplied by the agent that exerts force F on the piston). Note that if the fluid has pulled away from the walls of the needle, the piston and the syringe, there is no fluid friction at these surfaces, and hence no additional equations involving the fluid viscosity.

To obtain a solution to the present problem, in which the fluid leaves the syringe via two flow paths, we suppose that the *vena contracta* does not occur, such that the fluid fills the needle and the annulus around the piston, and there is fluid friction at the walls of these regions.[1] Then, there are only three unknowns, P, V' and V'', which can be determined

[1]This contrasts with the related case of a shock absorber (hydraulic brake) [2], in which there is only a single flow path, and the *vena contracta* must be taken into account.

using mass conservation, and Poiseuille's law for flow with friction in the needle and in the annulus.

The equation of mass conservation is,

$$Q = VA_s = V'A' + (V + V'')A'' \approx V'A' + V''A'' = Q' + Q'', \tag{1}$$

where $A'' = A_s - A_p = \pi R_s^2 - \pi R_p^2 \ll A_s$ is the cross sectional area of the annulus, and $V + V''$ is the velocity of the fluid in the annulus relative to the moving piston, taking V'' to be the velocity of the fluid in the annulus relative to the syringe. For a narrow annulus, $V'' \gg V$, and we will neglect the small correction due to V in the following.

Poiseuille's law [3]-[8] for flow of a fluid with dynamic viscosity η in the (circular) needle of radius R', area $A' = \pi R'^2$, and length L' is,

$$Q' = V'A' = \frac{\pi R'^4 (P - P_0)}{8\eta L'}\,, \tag{2}$$

where P_0 is the atmospheric pressure at the outlet of the needle.[2]

The flow Q'' in the annulus (of length L) is a variant of so-called Taylor-Couette [10] flow, where the two cylindrical surfaces have a relative axial velocity, rather than being in relative rotation. For simplicity, we suppose that flow in the annulus is sufficiently well approximated by planar Poiseuille flow in a rectangular slot of thickness $t = R_s - R_p$ and width $w = 2\pi R_s \gg t$. In this, we expect that for $w \gg t$ the flow rate will be proportional to the width w of the channel. Then, by dimensional analysis, the dependence on the thickness t of the channel must be t^3.[3] We infer that the flow rate is $Q'' \approx t^3 w \Delta P/\eta L$, and in the limit that $R_p = 0$ we must have $Q'' = \pi R_s^4 (P - P_0)/8\eta L$ following eq. (2). Hence, a simple approximation is that[4]

$$Q'' = V''A'' \approx \frac{t^3 w (P - P_0)}{16\pi\eta L} = \frac{\pi (R_s - R_p)^3 R_s (P - P_0)}{8\eta L}\,. \tag{3}$$

Using eqs. (2)-(3) in eq. (1), the pressure P inside the syringe is

$$P \approx P_0 + \frac{8\eta V A_s L L'}{\pi [L R'^4 + L' R_s (R_s - R_p)^3]}\,. \tag{4}$$

The (average) flow velocities are then,

$$V' \approx V \frac{A_s}{A'} \frac{L R'^4}{[L R'^4 + L' R_s (R_s - R_p)^3]}\,, \qquad V'' \approx V \frac{A_s}{A''} \frac{L' R_s (R_s - R_p)^3}{[L R'^4 + L' R_s (R_s - R_p)^3]}\,. \tag{5}$$

[2] Poiseuille's law may have been first derived in [9].

[3] The impressive result that the dependence of the flow rate Q'' on the small thickness t is t^3 rather than t or t^2 illustrates that fluid dynamics is not intuitive.

[4] For planar Poseiulle flow, the 8 in eq. (3) is 12. See, for example, p. 843 of [11]. For annular Poiseuille flow, the "exact" result is $Q'' = \pi[R_s^4 - R_p^4 - (R_s^2 - R_p^2)^2/\ln(R_s/R_p)]/8\eta L \approx \pi(R_s - R_p)^3 R_p (P - P_0)/6\eta L$ for small $R_s - R_p$. See, for example, eq. (3-51), sec. 3-3.3 of [12], or eq. (3.8), sec. 3.1 of [13]. The "exact" form of Q'' goes to $\pi R_s^4/8\eta L$ as $R_p \to 0$.

2.1 $L,\ L' \to 0$

If the lengths L and L' of the piston/annulus and of the needle are very short, friction in the annulus and in the needle become negligible. In this case the fluid flow will exhibit the *vena contracta* [1] at both the annulus and the "needle" (which is now simply a circular aperture in the end wall of the syringe).

For a solution when viscosity is neglected, we estimate the areas of the flow in the annulus and in the "needle" to be 1/2 their geometrical areas, as holds approximately in many cases. And, since the work done on the fluid is the kinetic energy of the fluid (when viscosity and gravity are neglected), we have that (for $V \gg V''$),

$$\begin{aligned} F_{\rm total} &= FV - P_0\frac{A'}{2}V' - P_0\frac{A''}{2}V'' = PA_pV - P_0\frac{V'A' + V''A''}{2} \\ &= \frac{d\,{\rm KE}_{\rm fluid}}{dt} \approx \frac{\rho V'A'}{2}\frac{V'^2}{2} + \frac{\rho V''A''}{2}\frac{V''^2}{2} = \frac{\rho}{4}(V'^3A' + V''^3A''), \end{aligned} \tag{6}$$

noting that the mass of fluid accelerated per second from rest in the syringe to velocity V' in the "needle" is approximately $\rho V'A'/2$, and that accelerated each second to velocity V'' in the annulus is $\rho V''A''/2$, where A' and A'' are the geometric areas. Also, the time rate of change of momentum of the fluid is,

$$\begin{aligned} \frac{d\,p_{\rm fluid}}{dt} \approx \frac{\rho V'A'}{2}V' - \frac{\rho V''A''}{2}V'' &= F_{\rm total} = F + P_0A'' - P(A_s - A') - P_0A' \\ &= (P - P_0)(A' - A''). \end{aligned} \tag{7}$$

Finally, mass conservation, eq. (1), implies that,

$$VA_s \approx \frac{V'A' + V''A''}{2}. \tag{8}$$

We now have three equations for the three unknowns P, V' and V'', so in principle these unknowns are determined. However, eq. (6) is a cubic equation so the solution is algebraically intricate, and we do not give it here.[5]

The special case when $A' = 0$ (no needle) is the same as the hydraulic brake considered in sec. 2.2 of [2]. The case when $A'' = 0$ corresponds to a syringe without a leak (and with a zero-length "needle"), for which both eqs. (6) and (7) give $P = P_0 + \rho V'^2/2$ (for $A_p = A_s = A$ and mass conservation $VA = VA'/2$), as also follows from Bernoulli's equation[6] (which does apply to this simple case, unlike the cases where $A' = 0$ and the present example where both A' and A'' are nonzero).

Thanks to Edward Yang for pointing out that the approximation of eq. (3) was poor in a previous version of this note.

[5] Surprisingly, the solution for the case of nonzero lengths of the needle and piston, when effects of viscosity must be considered, is analytically simpler than the case of zero length of the needle and piston, for which viscosity can be ignored.

[6] Bernoulli's equation applies in situations where the kinetic + potential energy of the fluid is conserved, and where the fluid flow is steady. See [14] for further discussion of this point.

References

[1] K.T. McDonald, *Vena Contracta*, (Feb. 16, 2005),
http://kirkmcd.princeton.edu/examples/vena_contracta.pdf

[2] K.T. McDonald, *Hydraulic Brake*, (June 26, 2014),
http://kirkmcd.princeton.edu/examples/brake.pdf

[3] G. Hagen, *Ueber die Bewegung des Wassers in engen cylindrischen Röhren*, Ann. d. Phys. **43**, 323 (1839), http://kirkmcd.princeton.edu/examples/fluids/hagen_ap_43_323_39.pdf

[4] J.L.M. Poiseuille, Recherches expérimentales sur Ie mouvement des liquides dans les tubes de très petits diamètres, Comptes Ren. Acad. Sci. **11**, 961, 1041 (1840), **12** 112 (1841), **16**, 60 1843, http://kirkmcd.princeton.edu/examples/fluids/poiseuille_cras_11_961_40.pdf

[5] J.L.M. Poiseuille, Recherches expérimentales sur Ie mouvement des liquides dans les tubes de très-petits diamètres, Mem. Acad. Roy. Sci. Inst. France **9**, 433 (1846),
http://kirkmcd.princeton.edu/examples/fluids/pouiseuille_marsif_9_433_46.pdf

[6] M. Brillouin, *Jean Léonard Marie Poiseuille*, J. Rheo. **1**, 345 (1930),
http://kirkmcd.princeton.edu/examples/fluids/brillouin_jr_1_345_30.pdf

[7] J. Pfnitzer, *Poiseuille and his law*, Anaesthesia **31**, 273 (1976),
http://kirkmcd.princeton.edu/examples/fluids/pfitzner_anaesthesia_31_273_76.pdf

[8] S.P. Sutera and R. Skalak, *The History of Poiseuille's Law*, Ann. Rev. Fluid Mech. **25**, 1 (1993), http://kirkmcd.princeton.edu/examples/fluids/sutera_arfm_25_1_93.pdf

[9] E. Hagenbach, *Ueber die Bestimmung der Zähigkeit einer Flüssigkeit durch den Ausfluss aus Röhren*, Ann. d. Phys. **109**, 385 (1860),
http://kirkmcd.princeton.edu/examples/fluids/hagenbach_ap_109_385_60.pdf

[10] G.I. Taylor, *Stability of a Viscous Liquid contained between Two Rotating Cylinders*, Phil. Trans. Roy. Soc. London **223**, 23 (1923),
http://kirkmcd.princeton.edu/examples/fluids/taylor_ptrsla_223_289_23.pdf

[11] C. Wassgren, *Planar Couette-Poiseuille Flow* (Dec. 26, 2016),
http://kirkmcd.princeton.edu/examples/fluids/wassgren_16.pdf
https://engineering.purdue.edu/~wassgren/teaching/ME30800/Examples/NS_Equations_Example_01.pdf

[12] F.M. White and J. Majdalani, *Viscous Fluid flow*, 4th ed. (McGraw-Hill, 2022),
http://kirkmcd.princeton.edu/examples/fluids/white_22_e4.pdf

[13] P.G. Drazin and N. Riley, *The Navier-Stokes equations: a classification of flows and exact solutions* (Cambridge U. Press, 2006),
http://kirkmcd.princeton.edu/examples/fluids/drazin_06.pdf

[14] K.T. McDonald, *Is Bernoulli's Equation Relativistically Invariant?* (Sept. 13, 2009),
http://kirkmcd.princeton.edu/examples/bernoulli.pdf