Orbital and Spin Angular Momentum of the Fields of a Turnstile Antenna

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1 Problem

A “turnstile” antenna [1, 2] consists of a pair of linear dipole antennas oriented at 90° to each other, and driven 90° out of phase, as shown in Fig. 1.

The linear antennas could be either dipoles as shown in the figure, or simply monopoles. If a pair of loops antennas is used instead, the configuration is called an “eggbeater” antenna.

Consider the case that the length of the linear antennas is small compared to a wavelength, so that it suffices to characterize each antenna by its electric dipole \( \vec{p}_1, \vec{p}_2 \), where the magnitudes \( p_1 \) and \( p_2 \) are equal but their phases differ by 90°, the directions of the two moment differs by 90°, i.e., \( \vec{p}_1 \cdot \vec{p}_2 = 0 \), and \( \omega \) is the angular frequency.

Discuss the angular momentum of the fields of a turnstile antenna.

2 Solution

We consider a basic turnstile antenna whose component antennas lie in the \( x-y \) plane at a common point. Then, we can write the total electric dipole moment of the antenna system as,

\[
\vec{p} = p_0 e^{-i\omega t} = p_0 (\hat{x} + i\hat{y}) e^{-i\omega t},
\]

Figure 1: A “turnstile” antenna. From [2].

\( \hat{x} = \sin \theta \cos \phi \hat{r} + \cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi} \) and \( \hat{y} = \sin \theta \sin \phi \hat{r} + \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi} \) in a spherical coordinate system \((r, \theta, \phi)\).

1Note that \( \dot{\phi} \)
where \( p_0 \) is a real constant.

In Gaussian units (and in vacuum) these fields can be written as (see, for example, sec. 9.1 of [3]),

\[
\mathbf{E}(\mathbf{r}, t) = \left( \frac{k^2}{r} (\mathbf{\hat{r}} \times \mathbf{p}_0) \right) \times \mathbf{\hat{r}} + \left( \frac{i k}{r^2} + \frac{1}{r^3} \right) (3(\mathbf{p}_0 \cdot \mathbf{\hat{r}})\mathbf{\hat{r}} - \mathbf{p}_0) e^{i(kr - \omega t)},
\]

\[
\mathbf{B}(\mathbf{r}, t) = \left( \frac{k^2}{r} + \frac{i k}{r^2} \right) \mathbf{\hat{r}} \times \mathbf{p}_0 e^{i(kr - \omega t)}.
\]

where \( \mathbf{\hat{r}} = \mathbf{r}/r \) is the unit vector from the center of the dipole to the observer, \( c \) is the speed of light, and \( k = \omega/c \). The time-average Poynting vector is, in spherical coordinates \((r, \theta, \phi)\),

\[
\langle \mathbf{S} \rangle = \frac{c \text{Re}(\mathbf{E} \times \mathbf{B}^*)}{8\pi} = \frac{c k^4 (1 + \cos^2 \theta)}{8\pi r^2} \mathbf{\hat{r}} - \frac{c}{8\pi} \text{Re} \left[ \left( \frac{2k^2}{r^3} - \frac{i k^3}{r^4} - \frac{i k}{r^4} \right) (2(\mathbf{p}_0 \cdot \mathbf{\hat{r}})\mathbf{\hat{r}}_0^* + (3(\mathbf{p}_0 \cdot \mathbf{\hat{r}})(\mathbf{p}_0^* \cdot \mathbf{\hat{r}}) + 2p_0^2)\mathbf{\hat{r}}_0 \right].
\]

The time average density of field momentum is \( \langle \mathbf{S} \rangle /c^2 \), so the time-average density of field angular momentum is,

\[
\langle 1 \rangle = \mathbf{r} \times \frac{\langle \mathbf{S} \rangle}{c^2} = -\frac{1}{4\pi c} \text{Re} \left[ \left( \frac{2k^2}{r^3} - \frac{i k^3}{r^4} - \frac{i k}{r^4} \right) (\mathbf{p}_0 \cdot \mathbf{\hat{r}})\mathbf{\hat{r}}_0 \times \mathbf{\hat{r}}_0^* \right].
\]

We now restrict our attention to the far zone where the electromagnetic fields are,

\[
\mathbf{B} = \frac{k^2}{r} e^{i(kr - \omega t)} \mathbf{\hat{r}} \times \mathbf{p}_0, \quad \mathbf{E} = \mathbf{B} \times \mathbf{\hat{r}} = \frac{k^2}{r} e^{i(kr - \omega t)} \left( \mathbf{p}_0 - (\mathbf{p}_0 \cdot \mathbf{\hat{r}})\mathbf{\hat{r}} \right),
\]

whose components in spherical coordinates are,

\[
E_r = B_r = \mathbf{\hat{r}} \cdot \mathbf{B} = 0,
\]

\[
E_\theta = B_\phi = p_0 k^2 \frac{e^{i(kr - \omega t)}}{r} \cos \theta (\cos \phi + i \sin \phi),
\]

\[
E_\phi = -B_\theta = -p_0 k^2 \frac{e^{i(kr - \omega t)}}{r} (\sin \phi - i \cos \phi),
\]

noting that \( \mathbf{\hat{r}} \times \mathbf{\hat{r}} = \sin \phi \mathbf{\hat{\theta}} + \cos \theta \cos \phi \mathbf{\hat{\phi}} \), and \( \mathbf{\hat{r}} \times \mathbf{\hat{y}} = -\cos \phi \mathbf{\hat{\theta}} + \cos \theta \sin \phi \mathbf{\hat{\phi}} \). In the plane of the antenna, \( \theta = 90^\circ \), the electric field has no \( \theta \) component, and hence no \( z \) component; the turnstile radiation in the horizontal plane is horizontally polarized. In the vertical direction, \( \theta = 0^\circ \) or \( 180^\circ \), the radiation is circularly polarized. For intermediate angles \( \theta \) the radiation is elliptically polarized.

The magnitudes of the fields are,

\[
E = B = \frac{p_0 k^2}{r} \sqrt{1 + \cos^2 \theta},
\]

so the time-averaged radiation pattern is,

\[
\frac{d \langle P \rangle}{d\Omega} = \frac{c r^2}{8\pi} B^2 = r^2 \langle S_{\text{far}, r} \rangle = \frac{c p_0^2 k^4}{8\pi} (1 + \cos^2 \theta).
\]
The intensity of the radiation varies by a factor of 2 over the sphere. Compared to other simple antennas, this pattern is remarkably isotropic. The radiated power is greatest for \( \theta = 0 \) or \( 180^\circ \) in which directions the polarization is purely circular. The total time-average radiated power is,

\[
\langle P \rangle = \frac{2c p_0^2 k^3}{3}.
\]  

(12)

The time-average density of angular momentum in the far zone is,

\[
\langle l_{\text{far}} \rangle = \frac{k^3}{4 \pi c^2} \operatorname{Re} [i (p_0 \cdot \hat{r}) \hat{r} \times p_0^*] = -\frac{p_0^2 k^3}{4 \pi c r^2} \sin \theta \hat{\theta}.
\]  

(13)

This density flows radially outward at the speed of light, and we can speak of the rate of radiation of angular momentum in the far zone as,

\[
\frac{d}{dt} \langle l \rangle = cr^2 \langle l_{\text{far}} \rangle = -\frac{cp_0^2 k^3}{4 \pi} \sin \theta \hat{\theta} = \frac{cp_0^2 k^3}{4 \pi} \left( \sin^2 \theta \hat{z} - \sin \theta \cos \theta \hat{\rho} \right),
\]  

(14)

where \( \hat{\rho} \) is the radial unit vector in cylindrical coordinates \((\rho, \phi, z)\). Integrating over solid angle, we find that,

\[
\frac{d}{dt} \langle l \rangle = \frac{2cp_0^2 k^3}{3} \hat{z} = \frac{\langle P \rangle}{\omega} \hat{z},
\]  

(15)

which seems consistent with the notion that photons have angular momentum \( J = \hbar = U/\omega \).

### 2.1 Orbital and Spin Angular Momentum

The formalism that \( l = r \times S/c^2 \) implies that the density (and flow) of angular momentum has no radial component. However, we say that the fields along the \( z \)-axis are circularly polarized (and elliptically polarized in general). This suggests that there should be a description in which we can identify an angular momentum with a radial component for the fields along the \( z \)-axis.

A decomposition has been given in [4, 5] whereby the total field angular momentum can be written in three terms,

\[
\int 1 d\text{Vol} = \sum_i l_{\text{canonical}, i} + \int l_{\text{EM, orbital}} d\text{Vol} + \int l_{\text{EM, spin}} d\text{Vol},
\]  

(16)

\[
l_{\text{canonical}, i} = r \times p_{\text{canonical}, i}; \quad l_{\text{EM, orbital}} = r \times p_{\text{EM, orbital}}; \quad l_{\text{EM, spin}} = \frac{E_{\text{rot}} \times A_{\text{rot}}}{4\pi c},
\]  

(17)

and,

\[
p_{\text{EM, orbital}} = \frac{\sum_{j=1}^3 E_{\text{rot}, j} \nabla A_{\text{rot}, j}}{4\pi c},
\]  

(18)

where \( A_{\text{rot}} \) is the (gauge-invariant) rotational part of the (gauge-dependent) vector potential \( A \). That is, \( A_{\text{rot}} \) is the vector potential in the Coulomb gauge.

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2If the dipole moment \( p_0 \) were purely real, as for a small linear antenna, no angular momentum would be radiated.

3The canonical momentum \( p_{\text{canonical}, i} \) is nonzero only at the positions of charges \( e_i \), and does not contribute to angular momentum in the far zone.
The electric field is related to the Coulomb-gauge potentials by,

\[ \mathbf{E} = -\nabla V^{(C)} - \frac{1}{c} \frac{\partial \mathbf{A}^{(C)}}{\partial t} = -\nabla V^{(C)} - \frac{1}{c} \frac{\partial \mathbf{A}_{\text{rot}}}{\partial t} \approx ik\mathbf{A}_{\text{rot}} = \mathbf{E}_{\text{rot}}, \quad (19) \]

where the approximation hold in the far zone, noting that \(-\nabla V^{(C)}\) is the instantaneous electric dipole field, which falls off as \(1/r^3\). Then, in the far zone we have that the time-average spin-angular-momentum density, according to eq. (17), is,

\[ \langle l_{\text{EM,spin}} \rangle = \frac{\Re(\mathbf{E}_{\text{rot}} \times \mathbf{A}^{*}_{\text{rot}})}{8\pi c} = \frac{\Re(i\mathbf{E} \times \mathbf{E}^*)}{8\pi ck} = -\frac{\Im(\mathbf{E} \times \mathbf{E}^*)}{8\pi ck} \]

\[ = -\frac{p_0^2k^3}{8\pi cr^2} \Im \left[ \left( \mathbf{p}_0 \times \mathbf{p}^*_0 \right) \times \left( (\mathbf{r} \times \mathbf{p}_0^*) \times \mathbf{r} \right) \right] \]

\[ = -\frac{p_0^2k^3}{8\pi cr^2} \Im \left[ \mathbf{p}_0 \times \mathbf{p}^*_0 - 2i\Im \left[ (\mathbf{p}_0 \cdot \mathbf{r}) \mathbf{r} \times \mathbf{p}_0^* \right] \right] = \frac{p_0^2k^3}{4\pi cr^2} (\mathbf{\hat{z}} \sin \theta \mathbf{\hat{\theta}}) \]

\[ \langle l_{\text{EM,orbital}} \rangle = \mathbf{r} \times \langle \mathbf{p}_{\text{EM,orbital}} \rangle = \mathbf{r} \times \frac{\Re \left( \sum_{j=1}^{3} E_{\text{rot},j} \nabla A^{*}_{\text{rot},j} \right)}{8\pi c}. \quad (21) \]

We expect that this angular momentum density to vary as \(1/r^2\), so we must evaluate \(\langle \mathbf{p}_{\text{EM,orbital}} \rangle\) to order \(1/r^3\), which requires keeping terms in \(\mathbf{E}\) and \(\mathbf{A}_{\text{rot}}\) to order \(1/r^2\).

However, neither \(\langle l_{\text{far}} \rangle\) of eq. (13) nor \(\langle l_{\text{EM,orbital}} \rangle\) has a radial component, so that \(\langle l_{\text{EM,spin}} \rangle + \langle l_{\text{EM,orbital}} \rangle\) does not equal \(\langle l_{\text{far}} \rangle\).

That is, while the notion of a classical “spin” angular momentum is appealing as a precursor to a quantum analysis, it is not fully consistent in a classical-only view.

References


http://kirkmcd.princeton.edu/examples/QM/cohen-tannoudji_88.pdf