Spin-Orbit Coupling in the Earth-Moon System

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1 Problem

A prominent astronomical fact is that the Moon always shows the same face to the Earth. This means that the Moon rotates once about its axis each Earth month. It turns out that the "days" of Mercury and Venus are nearly equal to their respective "years,"[1] and that the periods of axial and orbital revolution are equal for most of the moons of Jupiter, Saturn, Uranus and Neptune. In 1879, George Darwin (son of Charles) proposed that this has come about due a coupling between the "day" and month/year via tidal friction [1]-[4] – resistance of the moon or planet to changes in shape induced by the $1/r^2$ variation of gravity of the body at the focus of its orbit, and that eventually the Earth day will equal one month.[2,3]

In this problem you should deduce a kind of existence proof that a spin-orbit coupling mechanism leads to changes of the "day" and the "month/year" such that these can eventually become equal.

For simplicity, consider a point satellite of mass m that revolves with orbital angular velocity ω around a planet of mass M in a nearly circular orbit of radius R. The planet rotates about its axis with "spin" angular velocity Ω, its moment of inertia about this axis is I, and this axis is perpendicular to the plane of the satellite's orbit.

Find expressions for the total angular momentum L of the system about its center of mass, and for the total (kinetic + potential) energy E. Eliminate R from these expressions to show that,

$$L = I\Omega + \frac{C}{\omega^{1/3}}\,, \qquad E = \frac{I\Omega^2}{2} - \frac{C\omega^{2/3}}{2}\,, \tag{1}$$

and deduce the value of C.

In general, the angular velocities ω and Ω are different. If $\omega \neq \Omega$ then tidal friction reduces the (kinetic + potential) energy E while conserving angular momentum. Show that there is a range of initial conditions such that eventually $\omega_0 = \Omega_0$.[4]

For the Earth-Moon system, Ω_E is decreasing with time. Give an expression for R as a function of Ω (and not ω) to show that R increases as Ω decreases. Then, by Kepler's law for the system, ω must be decreasing also.

Darwin noted that extrapolation of the above scenario into the past suggests there may have been a time when $R = R_E$ and the Earth and Moon were part of a single protoplanet.[5]

[1] Mercury's "day" is 2/3 of its "year."

[2] This hypothesis was first postulated by Kant (1754), pp. 6-9 of [5]. Kant's (verbal) argument is that if the Earth's day does not equal a month, then the tidal bulge caused by the Moon rotates with respect to the Earth and experiences tidal friction, which slows down the Earth's rotation until the day equals a month. The present problem is a slight mathematical elaboration of Kant's argument.

[3] That the length of a month is increasing seems to have been first noted by Halley (1695), p. 174 of [6].

[4] Hint: Consider the variable $x = C/\omega^{1/3}$ = orbital angular momentum.

[5] For a popular review, see [7], and also [8]. Nowadays, the so-called impact-origin hypothesis enjoys

2 Solution

The center of mass of the planet-satellite system are at distances,

$$r_M = \frac{m}{M+m} R, \qquad \text{and} \qquad r_m = \frac{M}{M+m} R \tag{2}$$

from the centers of these bodies, respectively, where $R = r_M + r_m$. The total angular momentum of the system (ignoring possible angular momentum associated with rotation of the satellite about its axis) in the rest frame of the center of mass of the system is the constant,

$$L = I\Omega + (Mr_M^2 + mr_m^2)\,\omega = I\Omega + \mu R^2 \omega, \qquad \omega = \frac{L - I\Omega}{\mu R^2}\,, \tag{3}$$

which provides a relation between the orbital angular velocity ω and the "spin" angular velocity Ω, where,

$$\mu = \frac{mM}{M+m} \tag{4}$$

is the reduced mass of the system. The total kinetic + potential energy of the system is,

$$E = KE + PE = \frac{I\Omega^2}{2} + \frac{(Mr_M^2 + mr_m^2)\omega^2}{2} - \frac{GMm}{R} = \frac{I\Omega^2}{2} + \frac{\mu R^2 \omega^2}{2} - \frac{GMm}{R}\,, \tag{5}$$

where G is Newton's gravitational constant.

The equations of motion,

$$M\ddot{\mathbf{r}}_M = -m\ddot{\mathbf{r}}_m = -\frac{GMm\mathbf{R}}{R^2}\,, \qquad \mathbf{R} = \mathbf{r}_M - \mathbf{r}_m, \tag{6}$$

lead readily for circular orbits to,

$$\frac{\mu R^2 \omega^2}{2} = \frac{GMm}{2R} = -\frac{PE}{2}\,, \qquad R^3 = \frac{GMm}{\mu\omega^2}\,. \tag{7}$$

The first form of eq. (7) is true in general for a $1/r^2$ attractive force according to the so-called virial theorem,[6] while the second form is Kepler's (3rd) law for the system when the orbits are nearly circular, as assumed here.

If we accept Kant's comment that the eventual effect of tidal friction is to make the (final) "spin" angular velocity Ω_0 "locked" to the final orbital angular velocity ω_0, at which time the bodies are distance R_0 apart, then conservation of angular momentum (3) and Kepler's 3rd law (7) that $R_i^3 \omega_i^2 = R_0^3 \omega_0^2$ suffice to determine ω_0 and R_0, according to,

$$L = I\Omega_i + \mu R_i^2 \omega_i = (I + \mu R_0^2)\omega_0 \approx \mu R_0^2 \omega_0 = \mu R_i^2 \omega_i \frac{\omega_0^{1/3}}{\omega_i^{1/3}}\,. \tag{8}$$

greater favor, although the issue remains unsettled. See, for example, [9, 10].

[6]See, for example, sec. 10 of [11].

Thus,

$$\omega_0 = \Omega_0 \approx \omega_i \left(1 + \frac{I\Omega_i}{\mu R_i^2 \omega_i}\right)^2 \qquad \text{for} \qquad (I \ll \mu R_0^2). \tag{9}$$

For the Earth-Moon system, this analysis predicts that the eventual day/month will be 48 present days, as first computed in sec. 276 of [12]. The only known example of a two-body system that has evolved to a final state in which both "days" equal their common "month" is Pluto and Charon.

To establish analytically that a final state can exist with $\omega_0 = \Omega_0$, we use eq. (7) in eq. (3) to write,[7]

$$L = I\Omega + \frac{(G^2\mu M^2 m^2)^{1/3}}{\omega^{1/3}} \equiv I\Omega + \frac{C}{\omega^{1/3}} \equiv I\Omega + x, \qquad \omega = \frac{C^3}{(L - I\Omega)^3}\,, \tag{10}$$

where x is the orbital angular momentum (which can be taken as positive by suitable choice of direction of the polar axis),

$$x = \mu R^2 \omega = \frac{C}{\omega^{1/3}} > 0, \qquad C = (G^2 \mu M^2 m^2)^{1/3}, \qquad \Omega = \frac{L - x}{I}\,. \tag{11}$$

Since the angular momentum is constant we can write,

$$0 = \frac{dL}{dx} = I\frac{d\Omega}{dx} + 1, \qquad \frac{d\Omega}{dx} = -\frac{3\omega^{4/3}}{C}\frac{d\Omega}{d\omega} = -\frac{1}{I}\,, \tag{12}$$

which implies that if Ω decreases then so does ω.

The energy (5) can now be written as,

$$E = \frac{I\Omega^2}{2} - \frac{GMm}{2R} = \frac{I\Omega^2}{2} - \frac{C\omega^{2/3}}{2} = \frac{I\Omega^2}{2} - \frac{C^3}{2x^2} = \frac{(L-x)^2}{2I} - \frac{C^3}{2x^2}\,. \tag{13}$$

Note that $E(x = 0) = -\infty$ and that $E(x = \infty) = \infty$, but that $E(x)$ is not necessarily a monotonic function. Taking the derivative of eq. (13), we have that,

$$\frac{dE}{dx} = -\frac{L-x}{I} + \frac{C^3}{x^3} = -\Omega + \omega = \frac{x}{I} + \frac{C^3}{x^3} - \frac{L}{I}\,. \tag{14}$$

Hence, if an equilibrium exists, where $dE(x_0)/dx = 0$, we have that $\Omega_0(x_0) = \omega_0(x_0)$, and the equilibrium "spin" and orbital angular velocities are "locked".

If we suppose that M represents the Moon and m represents the Earth, the above argument suggests that the period of rotation of the Moon about its axis should be equal to its orbital period once a certain kind of equilibrium was established in the past.[8] However, we can also suppose that M represents the Earth and m represents the Moon, in which case we anticipate that the Earth-Moon system can evolve until the Earth day equals one month, and both the Earth and the Moon present the same face to one another at all times.

[7] The remainder of this note follows [1].

[8] If the Moon consists of matter somehow ejected from the Earth, it is probable that the Moon was created with the lunar day equal to a month (at that early time).

The equilibrium, $dE/dx = 0$, exists in eq. (14) only if,

$$\mathrm{Min}\left(\frac{x}{I} + \frac{C^3}{x^3}\right) = 4\left(\frac{C}{3I}\right)^{3/4} < \frac{L}{I}, \tag{15}$$

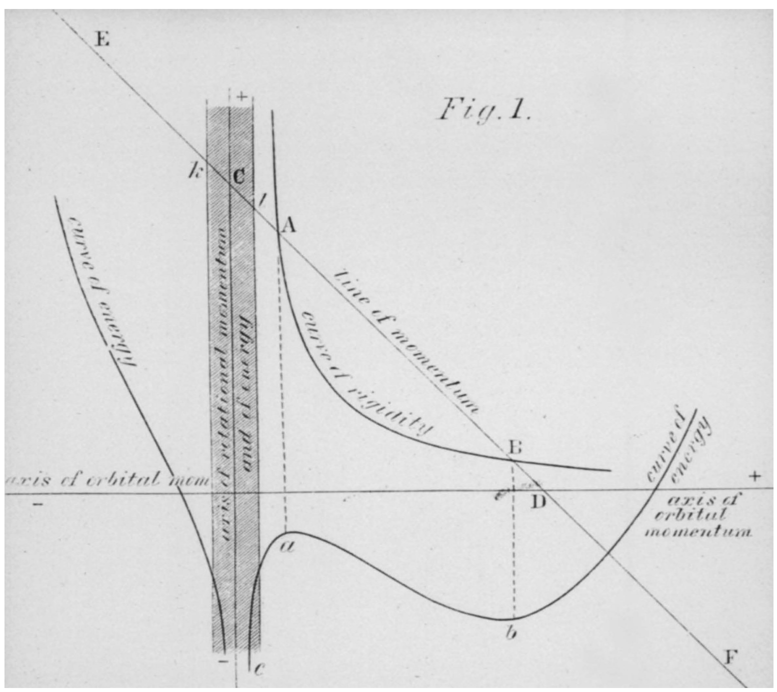
$$L = I\Omega_i + x_i > L_{\rm min} = 4I\left(\frac{C}{3I}\right)^{3/4} = \frac{4(3C^3I)^{1/4}}{3} = \frac{4x_{0,\rm min}}{3} > 0, \tag{16}$$

noting that the minimum occurs for $x_{0,\rm min} = (3C^3I)^{1/4}$. The requirement that L be positive (in the sense of the orbital angular momentum) means that if the "spin" angular momentum $I\Omega$ is opposite to the orbital angular momentum ($\mu R^2\omega = x$) and large, no equilibrium will exist.[9] Furthermore, if the evolution is to involve increasing orbital angular momentum x, as in the Earth-Moon system, the initial "spin" angular momentum $I\Omega_i$ must be a substantial fraction of the total for eventual equilibrium with $\omega_0 = \Omega_0$ to exist.

In greater detail, the equilibrium value x_0 of the orbital angular momentum is a root of the quartic equation obtained by setting eq. (14) to zero,

$$x^4 - Lx^3 + C^3I = 0. \tag{17}$$

When the condition (16) is satisfied, the so-called discriminant Δ of the quartic equation (17) is negative, which implies that there are two real roots and two complex roots.



The figure on the previous page, from [1], shows (among others) the lines labeled "curve of energy" which correspond to eq. (13); the curve on the left is for a case where no equilibrium exists, while for the curve on the right the stable equilibrium is at b, corresponding to the root x_0 of eq. (17), and the unstable equilibrium is at a, corresponding to the root x_1.

[9]In planetary systems where all objects have a common origin in an initial gas cloud, the sense of the angular momenta of all objects is typically the same.

When condition (16) is satisfied, the two real roots are,[10]

$$x_{0,1} = \frac{L}{4} - S \pm \sqrt{\frac{3L^2}{4} + \frac{L^3}{8S} - 4S^2}\,, \tag{18}$$

where,

$$S = \frac{1}{2}\sqrt{\frac{L^2}{4} + \frac{1}{3}\left(Q + \frac{81L_{\rm min}^4}{256Q}\right)}, \qquad Q = \left(\frac{729L_{\rm min}^4}{256}\right)^{1/3}\left(\frac{L^2 + \sqrt{L^4 - L_{\rm min}^4}}{2}\right)^{1/3}. \tag{19}$$

As tidal friction decreases the energy of the system, the equilibrium at x_0 (where $\omega_0 = \Omega_0$) can only be reached if the initial value of x is greater than x_1; otherwise the system evolves towards $x = 0$, which implies increasing ω, increasing Ω, and decreasing R until the two masses merge.[11] Hence, we deduce a condition on the initial orbital angular velocity ω_i for the existence of an equilibrium final state where $\omega_0 = \Omega_0$,

$$\omega_i < \frac{C^3}{x_1^3}\,. \tag{20}$$

If $\omega_i < C^3/x_0^3$, then as the energy decreases with time x decreases, ω increases, and R decreases; whereas if $C^3/x_0^3 < \omega_i < C^3/x_1^3$, then as the energy decreases x increases, ω decreases, and R increases with time. *The Earth-Moon system is of the latter type.*

Lastly, we equate the expressions for ω in eqs. (3) and (10) to obtain,

$$R = \frac{(L - I\Omega)^2}{\mu^{1/2}C^{3/2}}\,, \qquad \frac{dR}{d\Omega} = -\frac{2I(L - I\Omega)}{\mu^{1/2}C^{3/2}} = -\frac{2\mu^{1/2}I\omega R^2}{C^{3/2}}\,, \tag{21}$$

which implies that if Ω decreases then R increases. Taking the derivative of the first form of eq. (13), we find,

$$\frac{dE}{d\Omega} = I\Omega + \frac{GMm}{2R^2}\frac{dR}{d\Omega} = I(\Omega - \omega), \qquad d\Omega = \frac{dE}{I(\Omega - \omega)}\,. \tag{22}$$

Hence, as tidal friction reduces the kinetic + potential energy E, the “spin” angular velocity Ω decreases if $\Omega > \omega$, as holds for the Earth-Moon system. At the same time, the Earth-Moon distance R increases according to eq. (21), and the orbital angular frequency ω decreases according to eq. (12).

Additional discussion of spin-orbit coupling and “locking” is given in, for example, [18]-[20] in addition to [7]-[10].[12]

[10] We use the notation of https://en.wikipedia.org/wiki/Quartic_function

[11] This behavior has come to be called the satellite paradox, that the effect of an energy-dissipation mechanism in a (gravitational) two-body system can be to increase the kinetic energies of the bodies. For example, the effect of atmospheric drag on a satellite in a low orbit about the Earth is to increase the speed of the satellite as it slowly spirals inwards towards the Earth's surface [13]-[17]. Thus, the intuitive argument of Kant [5], as seconded by Lord Kelvin [12], that tidal friction lengthens the “day” and the “month/year” is not true in general.

[12] Refs. [18, 19] also considered effects of tilts of the axes of the spinning bodies with respect to the orbital plane.

This simplified model leaves open the question of the very early history of the Earth-Moon system, when the Earth day was much shorter than at present, and the Earth-Moon distance was comparable to the Earth's radius. The contemporary view that the Moon was ejected from the Earth during a collision with a large asteroid first gained prominence in [21].

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