The Fields Outside a Long Solenoid with a Time-Dependent Current

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Abstract

An instructive version of this well-known problem is the case of a current that is zero to $t < 0$ and varies as αt for $t > 0$. A generally excellent discussion of this case by Abbott and Griffiths features, however, a singularity in the fields at any point at the moment they first become nonzero. This singularity can be avoided by careful approximation, derived here using expressions for time-dependent fields rather than potentials. The result is that while the fields assume a quasistatic character for long times after the current has started to flow they include a small amount of radiation at short times. Such an effect was observed in a simple experiment involving a kitchen appliance.

1 Introduction

Electric fields outside a long solenoid with a changing current were first detected experimentally by Oliver Lodge in 1889 [1],[1] who presented a quasistatic derivation of the effect based on the integral form of Faraday's law,

$$\mathcal{E}mf = 2\pi r E = -\frac{\dot{\Phi}_B}{c} \tag{1}$$

in Gaussian units for a loop of radius r about an infinite solenoid of radius a along the z-axis that carries current $I = \alpha t$ per unit length. Then, Ampère's law was used to deduce the instantaneous magnetic field inside the solenoid as $B_z = 4\pi I/c$, and the magnetic field outside the solenoid was neglected in the evaluation of the magnetic flux Φ_B. The resulting fields outside the solenoid are,

$$E = -\frac{2\pi a^2 \alpha}{c^2 r}, \qquad \text{and} \qquad B \approx 0. \tag{2}$$

However, Lodge was very clear that an electric field outside the solenoid is to be expected because the number of magnetic field lines outside the magnet vary with time. Indeed, (he argued) magnetic field lines form closed loops only part of which lie within the solenoid. So, when the number of field lines through the core of the solenoid changes there must be a corresponding number of lines crossing any concentric cylinder external to the (long) solenoid. Then, according to Faraday, the movement of the magnetic field lines across the exterior cylinder will generate an $\mathcal{E}mf$, and consequently an electric field, around the cylinder. For all this, there must be a small magnetic field outside the solenoid.

[1] Recent articles in the Journal on this topic include refs. [2]-[7]. For the related example of a straight wire with a linearly rising current, see [8].

It has been noted [5] that the example of a linearly rising current which has persisted forever in an infinite solenoid is a special case in that Maxwell's equations are satisfied by an electric field as calculated above and a magnetic field that is zero outside the solenoid. This pedagogic quandary is reasonably avoided by noting that any real current began from zero at some finite time in the past [6].

Thus, a more meaningful example is that the current is nonzero only for $t > 0$. This case was treated by Abbott and Griffiths [2], who used the retarded potentials to deduce the following expressions for the fields $\mathbf{E}$ and $\mathbf{B}$ at the point $(r, 0, 0)$ in cylindrical coordinates (r, θ, z) outside a solenoid with linearly rising current, $I = \alpha t$ per unit length,

$$E_\theta = -\frac{2\pi a^2 \alpha}{c^2 r}\frac{ct}{z_0}, \qquad \text{and} \qquad B_z = -\frac{2\pi a^2 \alpha}{c^2 z_0}. \tag{3}$$

Here, $z_0 = \sqrt{(ct)^2 - r^2}$ is the position along the axis of the solenoid (measured from the point on the axis closest to the observer) such that the distance to the observer is ct. See Figure 1. It is easy to verify by direct differentiation that these results satisfy Maxwell's equations. For large times E_θ from eq. (3) becomes the same as that found in eq. (2), while B_z tends to zero. For ct just slightly greater than r, however, $E_\theta \approx B_z$ and these can be interpreted as radiation fields.

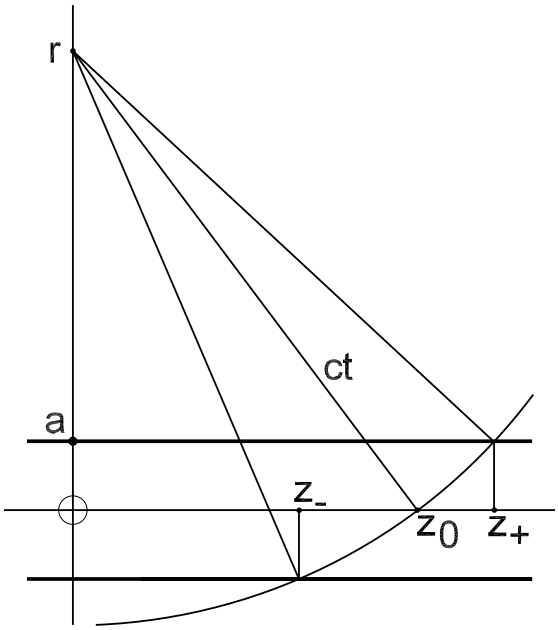

Figure 1: The fields of a solenoid of radius a concentric with the z-axis are observed at the point $(r, 0, 0)$ in a cylindrical coordinate system. At time t the observer receives radiation emitted at $t = 0$ from points z_+ and z_- on the near and far side of the solenoid, respectively, both of which are distance ct from the observer. The point on the axis at distance ct from the observer is labeled z_0.

However, the fields (3) are arbitrarily large at times t sufficiently close to, but larger

than, r/c. Yet, it is clear that for small enough time difference between ct and r the solenoid current appears to the observer as that due only to the portion of the surface nearest the observer. In this limit the effective current is at right angles to the axis of the solenoid and its magnitude is arbitrarily small (the time being arbitrarily close to zero at the source). This problem was also treated in the paper of Abbott and Griffiths where the corresponding radiation fields were found to be arbitrarily small, in agreement with reasonable expectation.

In this Note, I show how more careful approximation in the calculation of the fields of the solenoid for ct just greater than r leads to arbitrarily small, not arbitrarily large, values. However, like Abbott and Griffiths I assume an explicit form for the time dependence of the current. The calculation proves to be most delicate for times of order a/c after the current begins to flow, where a is the radius of the solenoid. It remains doubtful whether the assumed time-dependence of the current is a realistic approximation for times less than l/c, where l is the length of the solenoid and $l \gg a$. Departures of the current distribution at early times from the idealized form assumed here will more likely increase the radiation than decrease it; compare with problems 14.12 and 14.13 of the textbook of Jackson [9]. Hence, the experimental detection of radiation during the turn-on of an electric motor, as reported in the final section of this Note, suggests that the simplified model used here does contain much of the essential physics.

2 The Vector Potential

It is instructive to begin by considering the retarded vector potential for the solenoid (even though I will not carry this calculation to completion),

$$\mathbf{A}(\mathbf{x},t) = \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{x}', t' = t - R/c)}{R}\, d\mathbf{x}', \tag{4}$$

where $R = |\mathbf{R}|$ and $\mathbf{R} = \mathbf{x} - \mathbf{x}'$. In the present example the current density $\mathbf{J}$ is confined to the surface of the solenoid of radius a and has value $\alpha t\hat{\boldsymbol{\theta}}$ per unit length along the solenoid. The unit vector $\hat{\boldsymbol{\theta}}$ can be re-expressed as,

$$\hat{\boldsymbol{\theta}} = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta, \tag{5}$$

in terms of the unit vectors of a rectangular coordinate system.

The observer is outside the solenoid at $(x, y, z) = (r, 0, 0)$ with $r > a$. A typical point on the surface of the solenoid has cylindrical coordinates (a, θ, z) with corresponding cartesian coordinates $(a\cos\theta, a\sin\theta, z)$. Hence, the vector $\mathbf{R}$ has (x, y, z) components,

$$\mathbf{R} = (r - a\cos\theta, -a\sin\theta, -z), \tag{6}$$

and,

$$R = \sqrt{z^2 + r^2 + a^2 - 2ar\cos\theta} \tag{7}$$

The current is nonzero only for $t' > 0$, *i.e.*, for only $ct > R$. For a given angle θ on the solenoid there is a distance $z_{\max}$ such that this condition is satisfied for all $|z| < z_{\max}$.

The subtlety in the calculation is that the condition $t' > 0$ is not maintained for θ over the full range of 2π when ct is close to R. But, at each z there is a value $\theta_{\max}$ such that the condition holds for $|\theta| < \theta_{\max}$.

Then eq. (4) can be written as,

$$\mathbf{A}(r,0,0,t) = \frac{1}{c}\int_{-\theta_{\max}}^{\theta_{\max}} a\ d\theta \int_{-z_{\max}}^{z_{\max}} dz \frac{\alpha(t-R/c)(-\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta)}{R}. \tag{8}$$

The x-component of the integrand is odd in $\sin\theta$ and so vanishes on integration. Only component A_y survives,

$$A_y = \frac{a\alpha}{c}\int_{-\theta_{\max}}^{\theta_{\max}} \cos\theta\ d\theta \int_{-z_{\max}}^{z_{\max}} dz \left(\frac{t}{R} - \frac{1}{c}\right). \tag{9}$$

At this point it is tempting to make an approximation that proves not to be valid. Whenever $\theta_{\max} = \pi$ (which it is unless ct is very close to R) the integral of the term $\cos\theta/c$ vanishes. Thus, if we ignore the (hopefully) small contribution from the second term in the integrand from the region where $\theta_{\max} < \pi$ we could write,

$$A_y \approx \frac{a\alpha t}{c}\int_{-\theta_{\max}}^{\theta_{\max}} \cos\theta\ d\theta \int_{-z_{\max}}^{z_{\max}} \frac{dz}{R} \equiv \frac{a\alpha t}{c} f(r,t), \tag{10}$$

where $f(r,t)$ is the result of the remaining integration.

Before proceeding with the above approach (which leads to the results of Abbott and Griffiths) it is useful to develop a second method of calculation to aid in evaluating the merits of the proposed approximation.

3 Direct Calculation of the Fields

Expressions for the fields $\mathbf{E}$ and $\mathbf{B}$ in terms of time-dependent sources can be deduced by taking derivatives of the retarded potentials. These expressions have often been attributed to Jefimenko [10], although they appeared earlier in the textbook of Panofsky and Phillips [11, 12],

$$\mathbf{E} = \int \frac{[\rho]}{R^2}\hat{\mathbf{n}}\ d\mathbf{x}' + \frac{1}{c}\int \frac{[\dot{\rho}]}{R}\hat{\mathbf{n}}\ d\mathbf{x}' - \frac{1}{c^2}\int \frac{\left[\dot{\mathbf{J}}\right]}{R}\ d\mathbf{x}'. \tag{11}$$

where $\dot{\mathbf{J}} = \partial\mathbf{J}/\partial t$, $\hat{\mathbf{n}} = \mathbf{R}/R$ and,

$$\mathbf{B} = \frac{1}{c}\int \frac{[\mathbf{J}]\times\hat{\mathbf{n}}}{R^2}\ d\mathbf{x}' + \frac{1}{c^2}\int \frac{\left[\dot{\mathbf{J}}\right]\times\hat{\mathbf{n}}}{R}\ d\mathbf{x}', \tag{12}$$

Quantities in brackets, [ ], are to be evaluated at the retarded time $t' = t - R/c$.

Assuming that the wire of the solenoid remains neutral, the electric charge density ρ is zero (along with its time derivative). Hence, the electric field can be written as,

$$\mathbf{E} = -\frac{a\alpha}{c^2}\int_{-\theta_{\max}}^{\theta_{\max}} d\theta \int_{-z_{\max}}^{z_{\max}} dz \frac{-\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta}{R}. \tag{13}$$

As before, the x-component vanishes leaving,

$$E_y = -\frac{a\alpha}{c^2}\int_{-\theta_{\max}}^{\theta_{\max}} \cos\theta\, d\theta \int_{-z_{\max}}^{z_{\max}} \frac{dz}{R} = -\frac{a\alpha}{c^2} f(r,t), \tag{14}$$

where f is the same function introduced in eq. (10). No approximation has been made in deriving eq. (14) (other than the use of Maxwell's equations and a specified current distribution).

Of course, the electric field should be derivable from the vector potential via $\mathbf{E} = -(1/c)\partial\mathbf{A}/\partial t$, which according to eq. (10) implies,

$$E_y = -\frac{a\alpha}{c^2}\left(f(r,t) + t\frac{\partial f(r,t)}{\partial t}\right). \tag{15}$$

Comparison of eqs. (14) and (15) indicates that if the approximation in eq. (10) were valid then $\partial f/\partial t = 0$, and so the electric field should be constant in time.

Apparently the approximation used in deriving eq. (10) is not completely correct.

Either we should return to eq. (9) or continue on from eq. (14), both of which do not contain approximations. It appears to be more straightforward to continue with eq. (14). Thus, we have an example of how in practice direct evaluation of time-dependent fields can be as simple as, or simpler than, use of retarded potentials.

4 The Electric Field

To continue the evaluation of the integral in eq. (14) we anticipate that we will obtain results only for $r \gg a$, *i.e.*, for observers far from the solenoid.

By the symmetry of the problem, the electric field circulates about the solenoid and we interpret our calculation of E_y at $(r, 0, 0)$ as being E_θ at any (r, θ, z).

It is useful to introduce two additional distances, z_+ and z_-, corresponding to the smallest value of z at which $\theta_{\max} = 0$ and the largest value of z for which $\theta_{\max} = \pi$, respectively. See Figure 1. That is,

$$z_\pm = \sqrt{(ct)^2 - (r \mp a)^2} \approx \sqrt{z_0^2 \pm 2ar}, \tag{16}$$

recalling that $z_0 = \sqrt{(ct)^2 - r^2}$, and where the approximation neglects terms of order a^2. Radiation that leaves the solenoid at $t = 0$ from points $(r, \theta, z) = (a, 0, z_+)$ and (a, π, z_-) arrives at the observer at $(r, 0, 0)$ at time t. Equivalently, both source points are at distance ct from the observer.

For $|z| < z_-$ the angular integral in eq. (14) extends over 2π, corresponding to $\theta_{\max} = \pi$. So, we can split the integral into two ranges and reverse the order of integration in each to obtain,

$$E_\theta = -\frac{4a\alpha}{c^2}\int_0^{z_-} dz \int_0^{\pi} d\theta \frac{\cos\theta}{R} - \frac{4a\alpha}{c^2}\int_{z_-}^{z_+} dz \int_0^{\theta_{\max}} d\theta \frac{\cos\theta}{R} \tag{17}$$

In the approximation $r \gg a$ have from eq. (7),

$$\frac{1}{R} \approx \frac{1}{(r^2+z^2)^{1/2}}\left(1 + \frac{ar\cos\theta}{r^2+z^2}\right). \tag{18}$$

Using this in eq. (17) and performing the θ integrations we find,

$$E_\theta \approx -\frac{2\pi a^2 \alpha r}{c^2}\int_0^{z_-}\frac{dz}{(r^2+z^2)^{3/2}} - \frac{4a\alpha}{c^2}\int_{z_-}^{z_+} dz\left[\frac{\sin\theta_{\rm max}}{(r^2+z^2)^{1/2}} + \frac{ar}{(r^2+z^2)^{3/2}}\left(\frac{\theta_{\rm max}}{2}+\frac{\sin 2\theta_{\rm max}}{4}\right)\right]. \tag{19}$$

The first integral is standard but the second is awkward. For the latter we replace z by $\theta_{\rm max}$ as the variable of integration. First, note that for $z_- < |z| < z_+$,

$$(ct)^2 = z^2 + r^2 + a^2 - 2ar\cos\theta_{\rm max}, \tag{20}$$

so that,

$$\cos\theta_{\rm max} \approx \frac{(ct)^2 - r^2 - z^2}{2ar} = \frac{z_0^2 - z^2}{2ar} \approx \frac{z_0(z_0 - z)}{ar}. \tag{21}$$

Recalling eq. (16) we see that the second approximation in eq. (21) holds only for,

$$z_0 \gtrsim \sqrt{2ar}. \tag{22}$$

This approximation is distinct from our earlier approximation that $r \gg a$. We will have to examine separately the case $z_0 < \sqrt{2ar}$, which corresponds to the early times where the method of Abbott and Griffiths produced a singularity.

4.1 $z_0 > \sqrt{2ar}$

In this realm we have from eq. (21),

$$\sin\theta_{\rm max}\, d\theta_{\rm max} \approx \frac{z_0\, dz}{ar}. \tag{23}$$

With this, eq. (19) becomes,

$$E_\theta \approx -\frac{2\pi a^2\alpha}{c^2}\frac{z_-}{rct} - \frac{4a^2\alpha}{c^2}\frac{r}{z_0}\int_0^\pi d\theta_{\rm max}\frac{\sin^2\theta_{\rm max}}{(r^2+z^2)^{1/2}}, \tag{24}$$

where we have dropped terms of order a^3. In the remaining integral the factor $\sqrt{r^2+z^2}$ is close to ct. Similarly, $z_- \approx z_0$ for $z_0 > \sqrt{2ar}$. With these approximations we have,

$$E_\theta \approx -\frac{2\pi a^2\alpha}{c^2}\left(\frac{z_0}{rct} + \frac{r}{z_0 ct}\right) = -\frac{2\pi a^2\alpha}{c^2}\frac{ct}{rz_0}. \tag{25}$$

The result of the present approximations is the same as that found by Abbott and Griffiths (eq. (3)). Thus, after some effort we understand that their results hold only for $z_0 > \sqrt{2ar}$. This corresponds to times long enough that radiation has been received from the far side of the solenoid.

If we applied the approximations of Abbott and Griffiths [2] to the direct calculation of E_θ then we would obtain only the first term of eq. (24) but with z_- being called z_0. This is quite different from eq. (25) (and together with the corresponding result for the magnetic field does not satisfy Maxwell's equations), showing again that these approximations are not fully consistent.

It remains to make a calculation of the early times when radiation can be received only from the portions of the solenoid corresponding to $\theta_{\rm max} < \pi$.

4.2 $z_0 < \sqrt{2ar}$

From eq. (16) we deduce that z_- is defined only for $z_0 > \sqrt{2ar}$. Hence, for $z_0 < \sqrt{2ar}$ eq. (19) can be written as,

$$E_\theta \approx -\frac{4a\alpha}{c^2}\int_0^{z_+} dz \frac{\sin\theta_{\max}}{(r^2+z^2)^{1/2}} \approx -\frac{4a\alpha}{c^2}\frac{1}{r}\int_0^{z_+} dz \sin\theta_{\max}, \tag{26}$$

where for $z_0 < \sqrt{2ar}$ it proves preferable to approximate $\sqrt{r^2+z^2}$ by r rather than ct. From eqs. (16) and (20) we see that $\sin\theta_{\max} = \sqrt{(z_+^2 - z^2)/2ar}$. Hence,

$$E_\theta \approx -\frac{4a\alpha}{c^2}\frac{1}{r\sqrt{2ar}}\int_0^{z_+} dz\ \sqrt{z_+^2 - z^2} = -\frac{2\pi a^2\alpha}{c^2}\frac{z_+^2}{(2ar)^{3/2}}, \tag{27}$$

for early times when $z_0 < \sqrt{2ar}$ and correspondingly, $0 < z_+ = \sqrt{(ct)^2 - (r-a)^2} < 2\sqrt{ar}$. The approximate solutions (25) and (27) do not quite match at $z_0 = \sqrt{2ar}$ when the field is near its maximum, but are are good for times earlier or later than this.

5 The Magnetic Field

We use eq. (12) to evaluate the magnetic field. Recalling that $\mathbf{J} = \alpha t\ \hat{\boldsymbol{\theta}}$, we have $[\mathbf{J}] = \alpha(t - R/c)\ \hat{\boldsymbol{\theta}}$ and $\left[\dot{\mathbf{J}}\right] = \alpha\ \hat{\boldsymbol{\theta}}$, so

$$\mathbf{B} = \frac{\alpha t}{c}\int \frac{\hat{\boldsymbol{\theta}}\times\hat{\mathbf{n}}}{R^2}\ d\mathbf{x}'. \tag{28}$$

From eqs. (5) and (6) we have,

$$\hat{\boldsymbol{\theta}}\times\hat{\mathbf{n}} = \frac{1}{R}[z\cos\theta\ \hat{\mathbf{x}} + z\sin\theta\ \hat{\mathbf{y}} + (a - r\cos\theta\ \hat{\mathbf{z}}] \tag{29}$$

Thus the integrands of the x- and y-components of $\mathbf{B}$ are odd in z and so these components vanish on integration. The remaining component is,

$$B_z = \frac{\alpha t}{c}\int dz \int a\ d\theta\ \frac{a - r\cos\theta}{R^3} \approx \frac{a\alpha t}{c}\int \frac{dz}{(r^2+z^2)^{3/2}}\int d\theta\ \left(-r\cos\theta + a - \frac{3ar^2\cos^2\theta}{r^2+z^2}\right), \tag{30}$$

using approximation (18).

Again we split the integration over z into the intervals $[0, z_-]$ and $[z_-, z_+]$. On performing the θ integration and neglecting terms in a^3, we find,

$$B_z \approx \frac{4\pi a^2\alpha t}{c}\int_0^{z_-}\frac{dz}{(r^2+z^2)^{3/2}}\left(1 - \frac{3}{2}\frac{r^2}{r^2+z^2}\right) - \frac{4a^2\alpha}{c^2}\frac{r}{(ct)^2}\int_{z_-}^{z_+}\sin\theta_{\max}\ dz, \tag{31}$$

where on the interval $[z_-, z_+]$ we approximate $r^2 + z^2 \approx (ct)^2$.

As for the electric field, we evaluate the integrals separately for z_0 less than and greater than $\sqrt{2ar}$.

5.1 $z_0 > \sqrt{2ar}$

In this region, z_- is greater than zero and the first integral of eq. (31) can be found in tables. For the second integral we again change variables with the aid of eq. (23), after which the integration is elementary. This leads to,

$$B_z \approx -\frac{2\pi a^2\alpha}{c^2}\left(\frac{z_-(r+a)^2}{r^2(ct)^2}+\frac{r^2}{z_0(ct)^2}\right). \tag{32}$$

In the first term we may set $z_- \approx z_0$ and $r + a \approx r$, leading to,

$$B_z \approx -\frac{2\pi a^2\alpha}{c^2}\frac{1}{z_0}. \tag{33}$$

This is also the result of Abbott and Griffiths.

5.2 $z_0 < \sqrt{2ar}$

In this region. z_- is not defined. so only the second integral of eq. (31) contributes. In that, we approximate the factor $1/(ct)^2$ as $1/r^2$ and proceed as for the electric field:,

$$B_z \approx -\frac{4a\alpha}{c^2}\frac{1}{r}\int_0^{z_+}\sin\theta_{\max}\,dz \approx -\frac{2\pi a^2\alpha}{c^2}\frac{z_+^2}{(2ar)^{3/2}} = E_\theta. \tag{34}$$

As expected, the electric and magnetic radiation fields at early times have equal magnitudes, are mutually orthogonal, and are orthogonal to the line of sight to the closest point on the solenoid.

The radiation from the far side of the solenoid tends to cancel that from the near side. As time advances the z-coordinate from which radiation is received becomes more nearly the same at all azimuths around the solenoid and the cancellation becomes more perfect. The fields rise from zero until they reach a maximum near time $t = (r + a)/c$ corresponding to $z_0 = \sqrt{2ar}$ when radiation has first been received from the far side of the solenoid. The radiation fields die out rapidly thereafter. The small remaining time-dependent magnetic field is, however, sufficient to induce locally an electric field of the instantaneous quasistatic value (2). There is no need to invoke action at a distance, as might be required in a view that emphasizes only the quasistatic limit.

6 Radiated Power

The power radiated per unit length by the solenoid at time t_0 measured at the solenoid can be found from the Poynting vector, $\mathbf{S} = (c/4\pi)\mathbf{E}\times\mathbf{B}$, by integrating it over a large cylinder of radius r at time $t = t_0 + r/c$ [2],

$$P(t_0) = \lim_{r\to\infty} 2\pi r S(t_0 + r/c). \tag{35}$$

In this, $ct = ct_0 + r$ so $z_0 = \sqrt{(ct)^2 - r^2} \approx \sqrt{2rct_0}$ and $z_+ \approx \sqrt{2r(ct_0 + a)}$. The fields found above are written separately for z_0 less than or greater than $\sqrt{2ar}$, corresponding to

t_0 less than or greater than a/c, the time it takes radiation to moves across the radius of the solenoid. A subtlety: by defining the time t_0 at the solenoid as $t_0 = t - r/c$ in terms of time t at a distant observer, the radiation begins at $t_0 = -a/c$ since it first arrives at the observer at $t = (r - a)/c$.

Inserting eqs. (3) and (34) into (35) we find,

$$P(t_0) \approx \frac{\pi^2 a^3 \alpha^2}{c^3} \begin{cases} (1 + ct_0/a)^2, & -a/c < t_0 < a/c, \\ a/ct_0, & t_0 > a/c. \end{cases} \tag{36}$$

The total energy radiated up to time $t_0 > a/c$ is then,

$$U_{\rm rad} \approx \frac{\pi^2 a^4 \alpha^2}{c^4} \left(\frac{8}{3} + \ln \frac{ct_0}{a} \right). \tag{37}$$

It is interesting to compare this to the energy lost to Joule heating. If the solenoid coil has thickness b and resistivity ρ then the resistance of length l of the solenoid is $R = 2\pi a\rho/bl$. That is, the resistance varies inversely with length, regarding the turns of the coil in parallel. The current in length l at time t_0 is $I = \alpha t_0 l$, so the rate of Joule heating is,

$$I^2 R = \frac{2\pi a \rho \alpha^2 t_0^2 l}{b}, \tag{38}$$

which is proportional to length l. The total energy dissipated in heat per unit length up to time t_0 is then,

$$U_{\rm Joule} = \frac{2\pi a \rho \alpha^2 t_0^3}{3b} = \frac{2c\rho}{3\pi b} \frac{\pi^2 a^4 \alpha^2}{c^4} \left(\frac{ct_0}{a} \right)^3, \tag{39}$$

ignoring the tiny contribution from $-a/c < t_0 < 0$. In a typical metal $\rho \approx 10^{-6}$ ohm-cm, while 1 ohm $= 1/30c$ in Gaussian units. Thus the factor $2c\rho/3\pi b$ is about $10^{-6}/45\pi b$ for b in cm. A typical diameter of the coil wire is $b \approx 1/4.5\pi$ cm, so the factor is about 10^{-7}. Comparing eqs. (37) and (39) we see that the total energy lost to radiation is greater than that lost to heat until,

$$\left(\frac{ct_0}{a} \right)^3 \approx 10^7 \left(\frac{8}{3} + \ln \frac{ct_0}{a} \right), \tag{40}$$

corresponding to $t_0 \approx 450a/c$. For a solenoid with radius a of 1 cm this transition time is only about 15 nsec. This result tells us that the initial current in a solenoid need be linear with time only for a few nanoseconds for the analysis of this Note to be a good approximation to the transient electromagnetic effects. The currents in an L-R circuit are linear up to times of order L/R, so the present analysis could be applicable to many practical cases.

7 Discussion

The major qualitative result of the present analysis is the same as that of Abbott and Griffiths [2]: the fields outside a solenoid include electromagnetic radiation for a short time after the current starts to flow. Is this radiation in fact detectable?

Heald [3] notes that establishing a field inside a solenoid (or toroid) via an external power source requires lines of the Poynting vector to point from the power source into the solenoid. It could be that the corresponding flow of power through the electromagnetic field masks the transient radiation effect discussed above.[2] However, for a linearly rising current the energy stored in the solenoid varies quadratically with time, so the Poynting flux considered by Heald increases with time and is therefore reasonably distinct from the transient radiation effect considered here. Of course, the Poynting flux from the power source will have a turn-on transient that, in general, will have the character of radiation. It remains that not all the energy delivered from the power source ends up stored in the (quasi)static field of the solenoid; some energy escapes in the form of radiation.

In a different approach to the startup of the field of a solenoid, Protheroe and Koks [7] considered a double-wound solenoid with a small gap between the two windings. This model has the merit of being calculable in detail. For equal and opposite currents in the two windings the fields exist only between the windings. The transient fields propagate parallel to the axis of the solenoid from the end where the power source is located. The velocity of propagation is found to be $c/(2\pi an) \ll c$ where n is the number of windings per unit length and a is the solenoid radius. In this model there are no radiation fields (as well as no quasistatic field inside the inner solenoid). The authors argued that this result can be extended to a typical solenoid to which the power source is connected by a pair of leads that do not run close to the surface of the solenoid. I do not find this conclusion to be convincing. Rather, I find the argument of Heald [3] to be more representative of the general case; most of the energy in a typical solenoid enters at right angles to the axis rather than along the axis.

In view of these ambiguities, it is useful to follow the example of Lodge and perform an experiment.[3] Most of us have heard a noise pulse on a radio when some nearby appliance with an electric motor is switched off (due to the inductive spark). Is there also a noise pulse when the motor (a crude approximation to an infinite solenoid) first turns on? Indeed, when starting my electric can opener (model 752R, Rival Mfg. Co., St. Dalia, MO 65301) near a battery-powered radio tuned between stations, a noise pulse can be heard over the radio both when the device is turned on and off. There is essentially no noise during the steady operation of the can opener. I infer that the can opener indeed emits transient radiation when it is switched on, thereby confirming the spirit of the main argument of this Note. Thus the topic of this Note is another example of physics in the kitchen.

References

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[2] (August 10, 2020) It is now the view of this author that the Poynting vector should be identified with electromagnetic "radiation" [13].

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