1 Problem

A popular ride at amusement parks is the “slingshot,” in which two bungee cords of rest length \( l_0 \) and spring constant \( k \) are attached between two poles distance \( 2l \) apart and connected to mass \( m \). The mass is lowered by height \( H > 0 \) below the tops of the poles, and then released.

What is the maximum velocity of the mass?
What is the maximum height \( h \) above the tops of the poles reached by the mass? For this, suppose that \( l_0 = 0 \).
What are the frequencies of the normal modes of small oscillation of the system about equilibrium?

2 Solution

We assume that there is no energy dissipation in the bungee cords. Then for purely vertical motion along the \( z \)-axis, with \( z = 0 \) at the top of the poles, the energy is,

\[
E = \frac{mv^2}{2} + k \left( \sqrt{z^2 + l^2} - l_0 \right)^2 + mgz \\
= k \left( \sqrt{H^2 + l^2} - l_0 \right)^2 - mgH \\
= k \left( \sqrt{h^2 + l^2} - l_0 \right)^2 + mgh \\
= \frac{mv^2_{\text{max}}}{2}.
\] (1)

The maximum velocity occurs when the mass passes by \( z = 0 \), where,

\[
v_{\text{max}} = \sqrt{\frac{2k}{m} \left[ H^2 + 2l_0 \left( \sqrt{H^2 + l^2} - l \right) \right] - 2gH} \quad \rightarrow \quad \sqrt{\frac{2kH^2}{m} - 2gH} \quad \text{if} \quad l_0 = 0.
\] (2)

To find the maximum height \( h \) we equate the second and third lines of eq. (1), which leads to a quartic equation in \( h \) if \( l_0 > 0 \). To obtain a simple analytic result we suppose that \( l_0 = 0 \), in which case we find only a quadratic equation in \( h \),

\[
h^2 + \frac{mgh}{k} + \frac{mgH}{k} - H^2 = 0 = \left( h + H \right) \left( h - H + \frac{mg}{k} \right),
\] (3)
so that the maximum height is,

$$h = H - \frac{mg}{k}.$$  \hspace{1cm} (4)

The general motion is in all three coordinates $x$, $y$ and $z$, where we take the $x$-axis along the line connecting the tops of the poles. One normal mode involves purely vertical oscillations, and another is simple pendulum motion in the $y$-$z$ plane. The third normal mode is orthogonal to the first two, so should involve oscillation only in $x$.

For purely vertical motion,

$$m\ddot{z} = -mg - \frac{2kz}{\sqrt{z^2 + l^2}} \left(\sqrt{z^2 + l^2} - l_0\right).$$  \hspace{1cm} (5)

Again, an analytic description is much simpler if $l_0 = 0$. Then,

$$m\ddot{z} = -mg - 2kz,$$  \hspace{1cm} (6)

for which the equilibrium is at,

$$z_0 = -\frac{mg}{2k},$$  \hspace{1cm} (7)

and the angular frequency of small oscillations is,

$$\omega_1 = \sqrt{\frac{2k}{m}}.$$  \hspace{1cm} (8)

The second mode is simple pendulum motion in the $y$-$z$ plane with length $|z_0| = mg/2k$. The angular frequency of small oscillations for this mode is,

$$\omega_2 = \sqrt{\frac{g}{|z_0|}} = \sqrt{\frac{2k}{m}} = \omega_1.$$  \hspace{1cm} (9)

The third mode is for oscillations along the horizontal line with $y = 0$, $z = z_0$, for which the equation of motion is,

$$m\ddot{x} = -k \left(\frac{x}{\sqrt{(x-l)^2 + z_0^2}} \sqrt{(x-l)^2 + z_0^2} + \frac{x}{\sqrt{(2l-x)^2 + z_0^2}} \sqrt{(2l-x)^2 + z_0^2}\right) = -2kx.$$  \hspace{1cm} (10)

The angular frequency of small oscillations for this mode is,

$$\omega_2 = \sqrt{\frac{2k}{m}} = \omega_1 = \omega_2.$$  \hspace{1cm} (11)

All three modes have the same frequency when $l_0 = 0$, and the system is equivalent to mass $m$ being tied to the equilibrium point $(0,0,z_0)$ by a spring of zero length and constant $2k$. 

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