Charging a Capacitor via a Transient RLC Circuit

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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1 Problem

Discuss the time evolution of various forms of energy a series RLC circuit that is energized at time $t = 0$ by a battery of voltage $V$. Include consideration of radiated energy, supposing that the circuit has the form of a circular loop of radius $a$.

This problem relates to the question of whether a capacitor can be charged without loss of energy. As confirmed in sec. 2.1, if the capacitor is charged to voltage $V$ in a simple RC circuit, then the resistor dissipates energy equal to that eventually stored in the capacitor. Heinrich [1] noted that this energy loss could be avoided if the battery is replaced by a variable power supply whose voltage is raised “slowly” to the desired value $V$. See also [2, 3]. If one capacitor is charged by another in a circuit with negligible resistance, there is again a loss of energy, to radiation in this case [4, 5, 6].

These analyses leave open the question of whether energy loss is inevitable whenever a capacitor is charged “quickly”. Show that a capacitor can be charged with only modest energy loss in an underdamped series RLC circuit if the battery is disconnected after $1/2$ cycle.

2 Solution

Energy flows from the battery into four forms: the $I^2R$ heating of the resistor, the electrostatic energy $U_C = CV^2/2$ that remains stored in the capacitor once the transient current has died out, the energy $U_L(t) = LI^2/2$ that is temporarily stored in the inductor while the current is nonzero, and the energy radiated away while the current in the circuit is changing. We assume that radius $a$ of the circuit is small compared to the wavelength of all significant frequency components of the radiation, so that the current $I$ is independent of position around the circuit and the radiation is well approximated as that associated with the magnetic dipole moment,

$$m(t) = \pi a^2 I(t),$$

namely,

$$\frac{dU_{\text{rad}}}{dt} = \frac{1}{6\pi c^4} \sqrt{\frac{\mu_0}{\epsilon_0}} m^2 = 2.4 \times 10^{-32} a^4 I^2.$$  

The Kirchhoff equation for the series RLC circuit is,

$$V = LI + IR + \frac{Q}{C},$$

$^{1}$The stored energy is $Q^2/2C \propto (\int I dt)^2$, while the energy dissipated is $\int I^2 R dt$. So if the current $I$ is smaller and lasts for a longer time, the stored energy can be the same but the energy dissipated will be less. To obtain a lower current in the circuit, the voltage applied during the characteristic time interval for energy dissipation must be smaller; hence the prescription to raise the voltage slowly.
whose time derivative is,

\[ 0 = LI + IR + \frac{I}{C}. \]  

We seek solutions of the form \(e^{-\alpha t}\), for which eq. (4) leads to the quadratic equation,

\[ L\alpha^2 - R\alpha + \frac{1}{C} = 0, \]

whose solutions are,

\[ \alpha_{1,2} = \frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = \frac{R}{2L} \pm i \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}. \]

The current in the circuit is zero at time \(t = 0\) when the battery is connected to the circuit (and it cannot jump instantaneously to a nonzero value because of the inductor). Hence, the total current in the circuit can be written as,

\[ I(t) = I_0(e^{-\alpha_1 t} - e^{-\alpha_2 t}) = 2I_0e^{-Rt/2L} \sinh \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} t = 2iI_0e^{-Rt/2L} \sin \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} t. \]

Just after the battery is connected, the voltage drops across the resistor and capacitor are still zero, so the initial voltage drop across the inductor is related by,

\[ V = LI(0) = LI_0(\alpha_2 - \alpha_1) = I_0 \sqrt{R^2 - \frac{4L}{C}} = iI_0 \sqrt{\frac{4L}{C} - R^2}. \]

We now consider the cases that \(R\) is larger or smaller than \(2\sqrt{L/C}\).

### 2.1 Overdamped Circuit: \(R > 2\sqrt{L/C}\)

In this case the current is given by,

\[ I(t) = \frac{V}{\sqrt{R^2 - \frac{4L}{C}}} (e^{-\alpha_1 t} - e^{-\alpha_2 t}) = \frac{V}{\sqrt{\frac{R^2}{4L^2} - \frac{1}{C}}} e^{-Rt/2L} \sinh \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} t. \]

The energy temporarily stored in the inductor at time \(t\) is,

\[ U_L(t) = \frac{LI^2}{2} = \frac{V^2L}{2} \frac{1}{\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}} e^{-Rt/L} \sinh^2 \sqrt{R^2/4L^2} - \frac{1}{LC} t. \]

For large resistance \(R\) the inductive energy reaches a maximum of \(U_{L,\text{max}} \approx UC\sqrt{4L/C} \gg UC\) at time \(t \approx (L/R) \ln(R^2C/L)\).

The power dissipated in the resistor is,

\[ \frac{dU_{\text{joule}}}{dt} = I^2R = \frac{V^2R}{R^2 - \frac{4L}{C}} (e^{-2\alpha_1 t} - 2e^{-(\alpha_1 + \alpha_2)t} + e^{-2\alpha_2 t}), \]
and the total energy dissipated after a long time is,

$$U_{\text{Joule}} = \frac{V^2 R}{R^2 - \frac{4L}{C}} \left(\frac{1}{2\alpha_1} - \frac{2}{\alpha_1 + \alpha_2} + \frac{1}{2\alpha_2}\right) = \frac{V^2 R}{R^2 - \frac{4L}{C}} \left(\frac{RC}{2} - \frac{2L}{R}\right) = \frac{CV^2}{2} = U_C, \quad (12)$$

where $U_C = CV^2/2$ is the energy stored in the capacitor at large time $t$.

The radiated power is obtained from eqs. (2) and (9),

$$\frac{dU_{\text{rad}}}{dt} = 2.4 \times 10^{-32} a^4 \frac{V^2}{R^2 - \frac{4L}{C}} \left(\frac{\alpha_1^2}{2} - 2\alpha_1\alpha_2 e^{-(\alpha_1+\alpha_2)t} + \frac{\alpha_2^2}{2} e^{-2\alpha_2 t}\right), \quad (13)$$

and the total radiated power after a long time is,

$$U_{\text{rad}} = 2.4 \times 10^{-32} a^4 \frac{V^2}{R^2 - \frac{4L}{C}} \left(\frac{\alpha_1}{2} - \frac{2\alpha_1\alpha_2}{\alpha_1 + \alpha_2} + \frac{\alpha_2}{2}\right) = 2.4 \times 10^{-32} a^4 \frac{U_C}{RLC}. \quad (14)$$

In principle the radiated energy can become large if the inductance is very small such that the second derivative $\ddot{I}$ becomes very large. However, the inductance of a loop of radius $a$ made of wire of radius $b$ is $L \approx \mu_0 a \ln(a/b)$, so the radiated power is bounded by,

$$U_{\text{rad}} \lesssim 3 \times 10^{-38} a^5 \ln \frac{1}{b} \frac{1}{RC} U_C, \quad (15)$$

(in SI units). In any practical, transient $RLC$ circuit the radiated energy is negligible.

In sum, when a capacitor is charged via an overdamped $RLC$ circuit, as much energy is lost to Joule heating as ends up stored in the capacitor.

### 2.2 Underdamped Circuit: $R < 2\sqrt{L/C}$

In this case the current is given by,

$$I(t) = \frac{V}{i\sqrt{\frac{4L}{C} - R^2}} (e^{-\alpha_1 t} - e^{-\alpha_2 t}) = \frac{V}{\omega L} e^{-Rt/2L} \sin \omega t, \quad (16)$$

where,

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}. \quad (17)$$

The energy temporarily stored in the inductor at time $t$ is,

$$U_{\text{L}}(t) = \frac{LI^2}{2} = \frac{V^2}{2\omega^2 L} e^{-Rt/L} \sin^2 \omega t. \quad (18)$$

For small resistance $R$ the inductive energy reaches a maximum of $U_{L,\text{max}} \approx U_C$ at time $t \approx \pi / 2\omega \approx \pi \sqrt{LC}/2$.

The charge $Q(t)$ on the capacitor at time $t$ is,

$$Q(t) = \int_0^t I(t) \, dt = \frac{V}{\omega^2 L} \int_0^{\omega t} e^{-Rx/2\omega L} \sin x \, dx$$

$$= \frac{V}{\omega^2 L} \frac{1}{1 + R^2/4\omega^2 L^2} \left[1 - e^{-Rt/2L} \left(\frac{R}{2\omega L} \sin \omega t + \cos \omega t\right)\right]$$

$$= VC \left[1 - e^{-Rt/2L} \left(\frac{R}{2\omega L} \sin \omega t + \cos \omega t\right)\right]. \quad (19)$$
The energy \( U_C(t) \) stored in the capacitor at time \( t \) is,

\[
U_C(t) = \frac{Q^2(t)}{2C} = U_C \left[ 1 - e^{-Rt/2L} \left( \frac{R}{2\omega L} \sin \omega t + \cos \omega t \right) \right]^2.
\]  (20)

The power dissipated in the resistor is,

\[
\frac{dU_{\text{Joule}}}{dt} = I^2 R = \frac{V^2 R}{\omega^2 L^2} e^{-Rt/L} \sin^2 \omega t,
\]  (21)

and the energy \( U_{\text{Joule}}(t) \) dissipated in the resistor up to time \( t \) is,

\[
U_{\text{Joule}}(t) = \frac{V^2 R}{\omega L^2} \int_0^t e^{-Rx/\omega L} \sin^2 x \, dx
= U_C \left[ 1 - e^{-Rt/L} \left( 1 + \frac{R^2 \sin^2 \omega t}{2\omega^2 L^2} + \frac{R}{2\omega L} \sin 2\omega t \right) \right].
\]  (22)

For large \( t \) the energy dissipated equals the energy stored. However, the battery could be disconnected from the circuit whenever the current is zero, i.e., at \( t = n\pi/\omega \). In particular, if the battery were disconnected at time \( t = \pi/\omega \), we would have,

\[
\frac{U_{\text{Joule}}(\pi/\omega)}{U_C(\pi/\omega)} = \frac{1 - e^{-\pi R/\omega L}}{1 + e^{-\pi R/2\omega L}} \approx \frac{\pi R}{2\sqrt{L/C}},
\]  (23)

where the approximation holds for small resistance \( R \). That is, the capacitor can be charged with only small loss of energy to Joule heating by use of a large \( L \), small \( R \), and connecting the battery for only 1/2 of a (damped) cycle. As a bonus, the resulting voltage on the capacitor is nearly twice that of the battery.

When \( R \ll \sqrt{L/C} \) the second time derivative of the current is,

\[
\ddot{I}(t) \approx \frac{V\omega}{L} e^{-Rt/2L} \sin \omega t.
\]  (24)

The radiated power is obtained from eqs. (2) and (24),

\[
\frac{dU_{\text{rad}}}{dt} \approx 2.4 \times 10^{-32} a^4 \frac{V^2 \omega^2}{L^2} e^{-Rt/L} \sin^2 \omega t,
\]  (25)

and the total radiated power up to time \( t \) is,

\[
U_{\text{rad}}(t) \approx 2.4 \times 10^{-32} a^4 \frac{U_C}{R L C} (1 - e^{-Rt/L}).
\]  (26)

Then,

\[
U_{\text{rad}}(\pi/\omega) \approx 2 \times 10^{-32} a^4 \frac{\pi U_C}{L \sqrt{L/C}} \ll U_C.
\]  (27)

Again, the radiation in this transient \( RLC \) circuit is negligible.

In sum, while a capacitor that is charged for long times in an underdamped \( RLC \) circuit stores only as much energy as is lost to Joule heating, if the battery is disconnected after 1/2 cycle, the stored energy can be large compared to the energy lost to heat and radiation.
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References


