Self-Induced Transparency

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1 Problem

Show that a distortionless electromagnetic pulse (a soliton) can propagate with its magnetic field at right angles to a steady magnetic field that is applied to a magnetic medium, the direction of propagation being perpendicular to both of these fields.

Deduce the equations of motion for the magnetization $\mathbf{M} = N\mathbf{m}$ of a medium that consists of N permanent dipoles $\mathbf{m}$ (with angular momentum $\mathbf{L} = \Gamma\mathbf{m}$) per unit volume when the medium is immersed in a magnetic field $\mathbf{B}$. Consider the specific example of a static magnetic field $B_0\hat{\mathbf{x}}$ and a pulse $B_y(z - vt)\hat{\mathbf{y}}$.

The physical picture is that the magnetic field B_y of the pulse precesses the dipoles in the x-z plane by exactly 360° as the pulse passes, restoring the medium to its initial condition – in which the dipoles are lined up with the static field $B_0\hat{\mathbf{x}}$.

The solution can be deduced considering the pulse area function defined by,

$$A(z, t) = \Gamma \int_{-\infty}^{t} B_y(z, t')\, dt'. \tag{1}$$

Show that if $B_0 = 0$ then a formal solution for the behavior of the medium is $M_x = M_0 \cos A$, $M_z = M_0 \sin A$, supposing that $M_x(t = -\infty) = M_0$, and $M_z(t = -\infty) = 0$ (even though B_x has been temporarily set to 0).

Generalize this solution to the case of nonzero B_0 (nonzero $\omega_0 \equiv \Gamma B_0$) by supposing that $M_z = F(\omega_0)M_0 \sin A$ where $F(0) = 1$ and $M_x = M_0[F(\omega_0)(\cos A - 1) + 1]$, where the latter form preserves the condition that $M_x = M_0$ when $A = 0$. Show that the equations of motion for the magnetization imply that area function obeys the Mathieu equation,

$$\ddot{A} = \frac{1}{\tau^2} \sin A, \tag{2}$$

where the function F is given by,

$$F(\omega_0) = \frac{1}{1 + \omega_0^2\tau^2}\,, \tag{3}$$

in terms of a constant τ that will prove to be a measure of the pulse width.

Solve eq. (2) by multiplying by $\dot{A}$, *etc.*, to show that a pulse solution with velocity v is,

$$A = 4 \tan^{-1}[e^{(t-z/v)/\tau}], \tag{4}$$

and the corresponding magnetic field pulse is,

$$B_y = \frac{2}{\Gamma\tau} \text{sech} \left(\frac{t - z/v}{\tau} \right) . \tag{5}$$

Also, give expressions for the components of the magnetization $\mathbf{M}(t)$.

2 Solution

The phenomenon of self-induced transparency for electromagnetic pulses in a magnetic medium was first analyzed and demonstrated in the laboratory by McCall and Hahn [1]. See also [2].

As in the related problem of wave amplification in a magnetic medium [3], we note that when a magnetic dipole $\mathbf{m}$ is subject to a magnetic field $\mathbf{B}$ it experiences a torque $\mathbf{m} \times \mathbf{B}$ that precesses the angular momentum $\mathbf{L} = \mathbf{m}/\Gamma$, where $\Gamma = m/L$ is the gryomagnetic ratio of the dipole. If the magnetic dipoles are electrons, then $\Gamma = e/2m_e c \approx 10^7$ Hz/gauss, where e and m_e are the charge and mass of the electron, and c is the speed of light. Thus,

$$\mathbf{m} \times \mathbf{B} = \frac{d\mathbf{L}}{dt} = \frac{1}{\Gamma}\frac{d\mathbf{m}}{dt} \,. \tag{6}$$

The equation of motion of a single moment is,

$$\frac{d\mathbf{m}}{dt} = \Gamma \mathbf{m} \times \mathbf{B}, \tag{7}$$

so the equation of motion for the magnetization $\mathbf{M} = N\mathbf{m}$ is therefore,

$$\frac{d\mathbf{M}}{dt} = \Gamma \mathbf{M} \times \mathbf{B}. \tag{8}$$

For a magnetic field $B_x\hat{\mathbf{x}} + B_y(t)\hat{\mathbf{y}}$, the components of eq. (8) are,

$$\frac{dM_x}{dt} = -\Gamma B_y M_z, \tag{9}$$

$$\frac{dM_y}{dt} = \Gamma B_x M_z \equiv \omega_0 M_z \tag{10}$$

$$\frac{dM_z}{dt} = \Gamma(B_y M_x - B_x M_y) = \Gamma B_y M_x - \omega_0 M_y, \tag{11}$$

where $\omega_0 \equiv \Gamma B_x$.

The hint is to consider the pulse area function,

$$A(z,t) = \Gamma \int_{-\infty}^{t} B_y(z,t')\ dt', \tag{12}$$

whose time derivatives are,

$$\dot{A} = \Gamma B_y, \qquad \ddot{A} = \Gamma \dot{B}_y. \tag{13}$$

First we consider the case that $\omega_0 = 0$, for which the equations of motion (9)-(11) reduce to,

$$\dot{M}_x = -\Gamma B_y M_z, \qquad M_y = \text{const.}, \qquad \dot{M}_z = \Gamma B_y M_x. \tag{14}$$

We readily find two solutions for M_x and M_z,

$$M_x = M_0 \sin A, \qquad M_z = -M_0 \cos A, \tag{15}$$

and,

$$M_x = M_0 \cos A, \qquad M_z = M_0 \sin A. \tag{16}$$

We seek a pulse solution for B_y, so $A(z, t = -\infty) = 0$ for any finite z, with the initial (and final) condition that the magnetization is aligned along the x axis, *i.e.*, $M_x(t = -\infty) = M_0$ and $M_z(t = -\infty) = 0$. Clearly, the solution (16) is of the desired character.

Turning to the case of nonzero B_0, *i.e.*, nonzero ω_0, we extrapolate solution (16) by supposing that,

$$M_z = F(\omega_0) M_0 \sin A, \tag{17}$$

where $F(0) = 1$. A related trial solution for M_x is,

$$M_x = M_0[F(\omega_0)(\cos A - 1) + 1], \tag{18}$$

which preserves the initial condition that $M_x(t = -\infty) = M_0$.

To verify these trial solutions, we differentiate eq. (11) and combine with eqs. (9)-(10) and (13),

$$\begin{aligned} \ddot{M}_z &= \Gamma \dot{B}_y M_x + \Gamma B_y \dot{M}_x - \omega_0 \dot{M}_y = \ddot{A} M_x - \Gamma^2 B_y^2 M_z - \omega_0^2 M_z \\ &= M_0 \ddot{A}[F(\cos A - 1) + 1] - M_0 \dot{A}^2 F \sin A - M_0 \omega_0^2 F \sin A. \end{aligned} \tag{19}$$

But also,

$$\ddot{M}_z = \frac{d^2}{dt^2}(F M_0 \sin A) = \frac{d}{dt}(F M_0 \dot{A} \cos A) = F M_0 \ddot{A} \cos A - F M_0 \dot{A}^2 \sin A. \tag{20}$$

Combing eqs. (19) and (20) we find,

$$\ddot{A} = \frac{\omega_0^2 F}{1 - F} \sin A = \frac{1}{\tau^2} \sin A, \tag{21}$$

where the constant τ is defined by,

$$\frac{1}{\tau^2} = \frac{\omega_0^2 F}{1 - F}\,, \qquad \text{and so} \qquad F(\omega_0) = \frac{1}{1 + \omega_0^2 \tau^2}\,, \tag{22}$$

which obeys $F(0) = 1$ as required.

It turns out that we do not need the most general solution of the Mathieu equation (21). It suffices to use the particular solution found on multiplying eq. (21) by $\dot{A}$,

$$\dot{A}\ddot{A} = \frac{1}{\tau^2} \dot{A} \sin A, \tag{23}$$

which integrates to,

$$\frac{\dot{A}^2}{2} = K - \frac{\cos A}{\tau^2}. \tag{24}$$

The pulse area A and its time derivative vanish at $t = -\infty$ for any finite z, so the constant of integration is $K = 1/\tau^2$. Thus,

$$\frac{\dot{A}^2}{2} = \frac{1 - \cos A}{\tau^2} = \frac{2}{\tau^2} \sin^2 \frac{A}{2}\,. \tag{25}$$

Taking the square root, we have,

$$\frac{\dot{A}}{2} = \frac{1}{\tau} \sin \frac{A}{2}\,, \tag{26}$$

or,

$$\frac{d(A/2)}{\sin(A/2)} = \frac{dt}{\tau}\,, \tag{27}$$

which integrates to,

$$\ln[\tan(A/4)] = \frac{t}{\tau} + K. \tag{28}$$

A clever trick is to evaluate the integration constant K at the time $t_0(z)$ such that half the pulse has arrived at position z: $A(t_0) = A_{\max}/2$. If we define $A_{\max}$ to be 2π, a compact solution is obtained. Thus,

$$A(t_0) = \pi, \qquad \tan \frac{A(t_0)}{4} = 1, \tag{29}$$

$$\ln[\tan(A(t_0)/4)] = 0 = \frac{t_0}{\tau} + K, \tag{30}$$

so $K = -t_0/\tau$. The solution (30) is now,

$$\ln[\tan(A/4)] = \frac{t - t_0}{\tau}\,, \tag{31}$$

so,

$$\tan \frac{A}{4} = e^{(t-t_0)/\tau}, \tag{32}$$

and,

$$A = 4 \tan^{-1}[e^{(t-t_0)/\tau}]. \tag{33}$$

Since we desire a solution that describes a traveling pulse with velocity v, we identify t_0 at point z as z/v, and write,

$$A = 4 \tan^{-1} f, \qquad \text{with} \qquad f = e^{(t-z/v)/\tau}. \tag{34}$$

We see that τ is the characteristic width of the pulse (although this is even clearer once we have deduced eq. (45).)

To evaluate the components of the magnetization $\mathbf{M}$, we need explicit forms for $\sin A$ and $\cos A$, which will also permit a confirmation that the solution (34) satisfies the Mathieu equation (21).

Since $\tan(A/4) = f$, we have,

$$\cos^2 \frac{A}{4} = \frac{1}{1+f^2}\,, \qquad \text{and} \qquad \sin^2 \frac{A}{4} = \frac{f^2}{1+f^2}\,. \tag{35}$$

Then,

$$\begin{aligned}
\sin A &= 2 \sin \frac{A}{2} \cos \frac{A}{2} = 4 \sin \frac{A}{4} \cos \frac{A}{4} \left(\cos^2 \frac{A}{4} - \sin^2 \frac{A}{4} \right) \\
&= 4 \sqrt{\frac{f^2}{1+f^2} \cdot \frac{1}{1+f^2}} \left(\frac{1-f^2}{1+f^2} \right) = 4 \frac{f(1-f^2)}{(1+f^2)^2} = 4 \frac{\frac{1}{f} - f}{(\frac{1}{f} + f)^2} \\
&= -2 \tanh \left(\frac{t - z/v}{\tau} \right) \operatorname{sech} \left(\frac{t - z/v}{\tau} \right).
\end{aligned} \tag{36}$$

Likewise,

$$\begin{aligned}\cos A &= 1 - 2\sin^2\frac{A}{2} = 1 - 8\sin^2\frac{A}{4}\cos^2\frac{A}{4} = 1 - \frac{8f^2}{(1+f^2)^2} = 1 - \frac{8}{(\frac{1}{f}+f)^2} \\ &= 1 - 2\ \mathrm{sech}^2\left(\frac{t - z/v}{\tau}\right). \end{aligned} \tag{37}$$

Also,

$$\dot{A} = \frac{4}{1+f^2}\dot{f} = \frac{4}{\tau}\frac{f}{1+f^2} = \frac{4}{\tau}\frac{1}{\frac{1}{f}+f} = \frac{2}{\tau}\mathrm{sech}\left(\frac{t-z/v}{\tau}\right), \tag{38}$$

noting that $\dot{f} = f/\tau$. Hence,

$$\ddot{A} = \frac{4}{\tau}\left(\frac{\dot{f}}{1+f^2} - \frac{2f^2\dot{f}}{(1+f^2)^2}\right) = \frac{4}{\tau^2}\frac{f(1-f^2)}{(1+f^2)^2} = \frac{\sin A}{\tau^2}, \tag{39}$$

in agreement with eq. (21).

The components of the time-dependent magnetization are obtained as follows:

$$\frac{M_x}{M_0} = 1 + F(\cos A - 1) = 1 - 2F\ \mathrm{sech}^2\left(\frac{t-z/v}{\tau}\right). \tag{40}$$

We see that $M_x(t = +\infty) = 1$, and $M_x(0) \approx -1$ for $\omega_0\tau \gg 1$ ($F \approx 1$).

$$\frac{M_z}{M_0} = F\sin A = -2F\ \tanh\left(\frac{t-z/v}{\tau}\right)\mathrm{sech}\left(\frac{t-z/v}{\tau}\right). \tag{41}$$

In the limit that $\omega_0\tau \ll 1$ ($F \approx 1$), we have,

$$\begin{aligned}\frac{M_x^2 + M_z^2}{M_0^2} &\approx 1 - 4\ \mathrm{sech}^2\left(\frac{t-z/v}{\tau}\right) + 4\ \mathrm{sech}^4\left(\frac{t-z/v}{\tau}\right) \\ &\quad + 4\ \tanh^2\left(\frac{t-z/v}{\tau}\right)\mathrm{sech}^2\left(\frac{t-z/v}{\tau}\right) \\ &= 1. \end{aligned} \tag{42}$$

The behavior of the magnetization in the x-z plane is essentially a single revolution about the y axis, beginning and ending with $M_x = M_0$, $M_z = 0$, and with $M_x = -M_0$ at the position of the peak of the traveling pulse.

To find M_y we use eq. (11) for $\dot{M}_z$ together with eqs. (13), (17)-(18), (22) and (38):

$$\begin{aligned}\omega_0 M_y &= \Gamma B_y M_x - \dot{M}_z = \Gamma B_y M_0[1 + F(\cos A - 1)] - F\dot{A}\cos A = \Gamma B_y M_0(1-F) \\ &= \Gamma B_y \omega_0^2\tau^2 F M_0 = \omega_0^2\tau^2 F M_0 \dot{A} = 2\omega_0^2\tau F M_0\ \mathrm{sech}\left(\frac{t-z/v}{\tau}\right). \end{aligned} \tag{43}$$

Thus,

$$M_y = 2\omega_0\tau F M_0\ \mathrm{sech}\left(\frac{t-z/v}{\tau}\right), \tag{44}$$

and,

$$B_y = \frac{M_y}{\Gamma\omega_0\tau^2 F M_0} = \frac{2}{\Gamma\tau}\mathrm{sech}\left(\frac{t - z/v}{\tau}\right) = \frac{2B_0}{\omega_0\tau}\mathrm{sech}\left(\frac{t - z/v}{\tau}\right). \tag{45}$$

This hyperbolic secant pulse propagates without distortion or attenuation, with the leading "edge" of the pulse putting energy into the medium by flipping the magnetic dipoles, and the trailing edge of the pulse extracting the same energy by flipping the dipoles back to their original position.

Since M_y is proportional to B_y, we see that the equations of motion (9)-(11) are nonlinear. The pulse height of the soliton wave (45) cannot be chosen arbitrarily, as in linear wave propagation, but must be inversely proportional to the pulsewidth τ. In the intersting limit that $\omega_0\tau \ll 1$, *i.e.*, where the pulse width is short compared to the Larmor precession period, the peak field strength of the pulse is large compared to the static field B_0 although the wave magnetization M_y is small compared to M_0.

From Faraday's law, we deduce that the electric field of the pulse is in the x direction, with,

$$E_x = \frac{v}{c}B_y. \tag{46}$$

Taking the curl of the fourth Maxwell equation (assuming the medium to have dielectric constant $\epsilon = 1$), we find,

$$\nabla^2\mathbf{H} - \nabla(\nabla\cdot\mathbf{H}) = \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}. \tag{47}$$

Since,

$$H_y = B_y - 4\pi M_y = (1 - 4\pi\Gamma\omega_0\tau^2 F M_0)B_y = \left(1 - 4\pi\omega_0^2\tau^2 F\frac{M_0}{B_0}\right)B_y, \tag{48}$$

the y component of eq. (47) tells us that,

$$\frac{v}{c} = \sqrt{1 - 4\pi\omega_0^2\tau^2 F\frac{M_0}{B_0}}. \tag{49}$$

The ratio of M_0 to B_0 in a magnetic medium can be as large as the effective permeability, *i.e.*, of order 10^3. In practice, not only is $\omega_0\tau \ll 1$, but also $\omega_0^2\tau^2 M_0/B_0 \ll 1$, so the soliton velocity v is approximately c.

References

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