Classical Diamagnetism and the Satellite Paradox

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1 Problem

In typical models of classical diamagnetism (see, for example, sec. 34-4 of [1] or sec. 11.5 of [2]) the change of the magnetic moment of an "atom" is deduced when an external magnetic field increases "slowly" from 0 to $\mathbf{B} = B\,\hat{\mathbf{z}}$. The "atom" consists of an electron of charge $-e$ and mass m that is in a circular orbit of radius r_0 in the x-y plane about a "fixed nucleus" of charge e' at the origin. It is tacitly assumed that the radius of the orbit remains r_0 at all times, although this is not consistent with the effect of perturbations on motion in a $1/r$ potential, even if radiation is ignored (as it must be in any classical model of a stable atom).[1]

Give a model of classical diamagnetism that includes the (small) effect of changes in the radius of the orbit as external magnetic field increases. Compare the present case to the so-called satellite paradox [4, 5, 6]. The latter is that the effect of atmospheric drag on a satellite in a low orbit about the Earth is to increase the speed of the satellite as it slowly spirals inwards towards the Earth's surface.

2 Solution

Aspects of this problem have been discussed in [7].

We suppose that at time $t = 0$ an external magnetic field, perpendicular to the (x-y) plane of the electron's orbit, turns on according to,

$$B(t) = \dot{B}\,t \qquad (t > 0), \tag{1}$$

where $\dot{B}$ is a constant. As a result, an external azimuthal electric field,

$$\mathbf{E}_{\rm ext}(r,t) = -\frac{r\dot{B}}{2c}\,\hat{\boldsymbol{\phi}} \qquad (t > 0), \tag{2}$$

is generated according to Faraday's law (in cylindrical coordinates (r, ϕ, z) with the "fixed nucleus" at the origin, and in Gaussian units).[2] Then, the external (Lorentz) force on the electron is,

$$\mathbf{F}_{\rm ext} = -e\left(\mathbf{E}_{\rm ext} + \frac{\mathbf{v}}{c}\times\mathbf{B}_{\rm ext}\right) = \frac{er\dot{B}}{2c}\,\hat{\boldsymbol{\phi}} - \frac{ev_\phi B}{c}\,\hat{\mathbf{r}} + \frac{e\dot{r}B}{c}\,\hat{\boldsymbol{\phi}}, \tag{3}$$

[1]Diamagnetism has recently been consider in another type of classical "atom" [3], consisting of an electron somehow confined to the surface of a sphere.

[2]Faraday's law actually only tells us that $\oint E_\phi\, r\, d\phi = -\pi r^2 \dot{B}/c$, so that $\langle E_\phi\rangle = -r\dot{B}/2c$ where the average is taken over the orbit. In general, the induced electric field includes another component in the x-y plane that is essentially uniform over the "atom", which has the effect of creating a small electric dipole moment for the "atom" while the magnetic field is changing. We ignore this tiny effect, which in any case vanishes once the magnetic field takes on a steady, nonzero value.

where $\mathbf{v} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\phi}\,\hat{\boldsymbol{\phi}}$, and we write $r\dot{\phi} = v_\phi$. As $v_\phi B = v_\phi t\dot{B}$, the first and second terms of eq. (3) are of similar order (and both are small compared to the Coulomb force $-ee'\,\hat{\mathbf{r}}/r^2$).

We suppose that the change in magnetic field is so slow that the resulting radial velocity $\dot{r}$ is negligible compared to the azimuthal velocity v_ϕ; hence, the third term is small compared to the first and second terms of eq. (3). However, the third term plays a role in insuring that "a magnetic field does no work". Thus, during time dt the electron moves distance,

$$d\mathbf{s} = \mathbf{v}\,dt = (\dot{r}\,\hat{\mathbf{r}} + r\dot{\phi}\,\hat{\boldsymbol{\phi}})\,dt \tag{4}$$

and the work done on it by the external fields is,

$$dW_{\rm ext} = \mathbf{F}\cdot d\mathbf{s} = -e\,\mathbf{v}\cdot\mathbf{E}_{\rm ext}\,dt = \frac{er^2\dot{\phi}\dot{B}}{2c}\,dt = \frac{erv_\phi\dot{B}}{2c}\,dt. \tag{5}$$

The equation of motion for the electron is,

$$\begin{aligned} m\mathbf{a} = m\left(\ddot{r} - r\dot{\phi}^2\right)\hat{\mathbf{r}} + m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\boldsymbol{\phi}} &= -e\left(\mathbf{E}_{\rm tot} + \frac{\mathbf{v}}{c}\times\mathbf{B}_{\rm ext}\right) \\ &= -\frac{ee'}{r^2}\,\hat{\mathbf{r}} + \frac{er\dot{B}}{2c}\,\hat{\boldsymbol{\phi}} - \frac{er\dot{\phi}B}{c}\,\hat{\mathbf{r}} + \frac{e\dot{r}B}{c}\,\hat{\boldsymbol{\phi}}, \end{aligned} \tag{6}$$

For slow changes in the magnetic field we neglect $\ddot{r}$ compared to $r\dot{\phi}^2$, so the radial equation of motion can be written as,

$$mr\dot{\phi}^2 \approx \frac{ee'}{r^2} + \frac{er\dot{\phi}B}{c} = \frac{mr_0^3\dot{\phi}_0^2}{r^2} + \frac{er\dot{\phi}B}{c}\,, \tag{7}$$

where r_0 and $v_0 = r_0\dot{\phi}_0$ are the radius and azimuthal velocity, respectively, of the circular orbit when $B = 0$.

The azimuthal equation of motion is,

$$m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) = \frac{er\dot{B}}{2c} + \frac{e\dot{r}B}{c}\,. \tag{8}$$

The terms $r\ddot{\phi}$ and $\dot{r}\dot{\phi}$ are of comparable magnitude. Multiplying eq. (8) by r, we can integrate to find the (mechanical) angular momentum,

$$L_z = mr^2\dot{\phi} = mr_0^2\dot{\phi}_0 + \frac{er^2B}{2c}\,. \tag{9}$$

Using eq. (9) for $\dot{\phi}$ in eq. (7), we find,

$$\frac{r_0^3}{r^3} - \frac{r_0^2}{r^2} \approx \frac{e^2rB^2}{4m^2c^2r_0\dot{\phi}_0^2}\,. \tag{10}$$

Assuming a solution of the form $r = r_0(1+\Delta)$, we obtain,

$$r \approx r_0\left(1 - \frac{e^2B^2}{4m^2c^2\dot{\phi}_0^2}\right) = r_0\left(1 - \frac{e^2r_0^2B^2}{4m^2c^2v_0^2}\right). \tag{11}$$

That the radius r of the perturbed orbit departs from r_0 by a term that is second order in the magnetic field B was mentioned in [7] without a detailed derivation.

Returning to eq. (9), the angular velocity is, to second order in B,

$$\dot{\phi} \approx \dot{\phi}_0 + \frac{eB}{2mc} + \frac{e^2 B^2}{2m^2 c^2 \dot{\phi}_0}\,, \tag{12}$$

and the azimuthal velocity is,

$$v_\phi \approx v_0 + \frac{e r_0 B}{2mc} + \frac{e^2 r_0^2 B^2}{2m^2 c^2 v_0}\,, \tag{13}$$

to the same order.

The motion (11)-(13) does not *have the character of that in the satellite paradox. For positive v_0, the energy of the classical "atom" is increased by the perturbing magnetic field. That is, $W_{\rm ext} > 0$ according to eq. (5), while the radius of the orbit decrease with time according to eq. (11), and the azimuthal velocity increases according to eq. (13). This is the behavior for negative, rather than positive, $\dot{W}_{\rm ext}$ in the satellite paradox [4, 5].*

To first order in the magnetic field B, eq. (13) is the same as obtained by supposing that the azimuthal force $-eE_\phi$ directly affects the azimuthal velocity (i.e., that eq. (8) can be approximated as $\dot{v}_\phi = e r_0 \dot{B}/2mc$), in contrast to the case of the satellite paradox where the change in azimuthal velocity is the negative of the naive expectation [4]. However, eq. (13) differs in second order from the result of this assumption, as needed to explain the energy balance to this order (see Appendix A).

Indeed, if one is willing to accept eq. (13) to first order as "obvious", then the radial equation of motion can be written as,

$$\frac{m v_\phi^2}{r} \approx \frac{ee'}{r^2} + \frac{e v_\phi B}{c} = \frac{m r_0 v_0^2}{r^2} + \frac{e v_\phi B}{c}\,, \tag{14}$$

which is a quadratic equation in $1/r$ with the approximate solution,

$$\frac{1}{r} \approx \frac{1}{r_0} - \frac{e^2 r_0 B^2}{4m^2 c^2 v_0^2}\,, \tag{15}$$

which leads to eq. (11) but with the opposite sign for the second term.

Coming at length to the issue of diamagnetism, we note that the orbital angular momentum is,

$$\mathbf{L} = m r v_\phi\, \hat{\mathbf{z}}, \tag{16}$$

The magnetic moment of the system is, following Larmor,

$$\boldsymbol{\mu} = -\frac{e}{2mc}\mathbf{L} = -\frac{e r v_\phi}{2c}\,\hat{\mathbf{z}} \approx -\frac{e r v_0}{2c}\,\hat{\mathbf{z}} - \frac{e^2 r_0^2}{4mc^2}\,\mathbf{B} + \frac{e^3 r_0^3 B^2}{8m^2 c^3 v_0}\,\hat{\mathbf{z}} \equiv \boldsymbol{\mu}_0 + \boldsymbol{\mu}_{\rm diamagnetic} + \mathcal{O}(B^2), \tag{17}$$

where,

$$\boldsymbol{\mu}_0 = -\frac{e r v_0}{2c}\,\hat{\mathbf{z}}, \qquad \text{and} \qquad \boldsymbol{\mu}_{\rm diamagnetic} = -\frac{e^2 r_0^2}{4mc^2}\,\mathbf{B} \tag{18}$$

as in the usual treatments [1, 2] which simply assume that $r = r_0$ always. From eq. (11) we see that this is a very good assumption within the context of a classical "atom" (so long as one ignores effects of radiation).

A Appendix: Energy Balance

The energy of the classical "atom" in a magnetic field $\mathbf{B} = B(t)\,\hat{\mathbf{z}}$ is,[3]

$$U = \frac{m(\dot{r}^2 + v_\phi^2)}{2} - \frac{ee'}{r}. \tag{19}$$

We ignore the contribution of $\dot{r}$ to the kinetic energy, and write,

$$\begin{aligned} U &\approx \frac{mv_\phi^2}{2} - \frac{mr_0 v_0^2}{r} \approx \frac{mv_0^2}{2}\left(1 + \frac{er_0 B}{mcv_0} + \frac{e^2 r_0^2 B^2}{4m^2c^2v_0^2}\right) - mv_0^2\left(1 + \frac{e^2 r_0^2 B^2}{4m^2c^2v_0^2}\right) \\ &= -\frac{mv_0^2}{2} + \frac{er_0 v_0 B}{2c} + \frac{e^2 r_0^2 B^2}{8mc^2} \end{aligned} \tag{20}$$

to second order in field strength B. Then, to this order,

$$\dot{U} \approx \frac{er_0 v_0 \dot{B}}{2c} + \frac{e^2 r_0^2 B\dot{B}}{4mc^2} = \frac{er_0 v_0 \dot{B}}{2c}\left(1 + \frac{er_0 B}{2mcv_0}\right) \approx \frac{erv_\phi \dot{B}}{2c} = \dot{W}_{\rm ext}, \tag{21}$$

recalling eqs. (5) and (13).

B Appendix: Why Doesn't the Satellite Paradox Hold for Classical Diamagnetism?

One answer is that "magnetic fields do no work".

The phenomenon of classical diamagnetism, as described by eqs. (11) and (13), contradicts the claim in [5] that the qualitative behavior seen in the satellite paradox (the velocity increases as the radius decreases, and *vice versa*) holds for any perturbation of motion in a $1/r$ potential. The argument in [5] is based on the claim that all relevant behavior of the system can be related to its energy, and that changes in the energy are due to the work done by the perturbing forces. However, since "magnetic fields do no work", they can lead to perturbations whose effect is not captured by arguments based on energy.

For completeness, we analyze the satellite paradox in a manner similar to that given in sec. 2.

Suppose the external force is purely a drag force,

$$\mathbf{F}_{\rm ext} = -Cv^p\,\mathbf{v} \approx -Cv_\phi^p(\dot{r}\,\hat{\mathbf{r}} + v_\phi\,\hat{\boldsymbol{\phi}}), \tag{22}$$

where p is a small non-negative number, perhaps 0 or 1, and C is the drag coefficient. Unlike the case of an "atom" in an external magnetic field, it suffices to consider only the case that v_ϕ is positive.

[3]Note that the energy, $-\boldsymbol{\mu}\cdot\mathbf{B} = ervB/2c \approx er_0v_0B/2c$, of the magnetic moment $\boldsymbol{\mu}$ in the external magnetic field $\mathbf{B}$ is already contained in the microscopic energy expression (19), which is also the energy in the Darwin approximation (see, for example, eq. (22) of [8]).

We again neglect $\ddot{r}$ compared to $r\dot{\phi}^2$ in the radial equation of motion,

$$\dot{\phi}^2 \approx \frac{\dot{\phi}_0^2 r_0^3}{r^3} - \frac{C\dot{r}v_\phi^p}{mr}, \qquad \text{or} \qquad v_\phi^2 \approx \frac{v_0^2 r_0}{r} - \frac{Cr\dot{r}v_\phi^p}{m}, \tag{23}$$

where v_0 is the azimuthal velocity of the (nearly) circular orbit at some reference radius r_0. The effect of the drag force is to decrease r, so that $\dot{r}$ is negative, and eq. (23) tells us that $v_\phi > v_0$ for all $r < r_0$. Thus, the radial equation of motion contains the qualitative nature of the satellite paradox, that the drag force increases the azimuthal velocity, rather than reducing it.

In the first approximation, we ignore the small radial component of the drag force, so that,

$$\dot{\phi}^2 \approx \frac{\dot{\phi}_0^2 r_0^3}{r^3}, \qquad \text{and} \qquad v_\phi \approx v_0 \left(\frac{r_0}{r}\right)^{1/2}. \tag{24}$$

The azimuthal equation of motion is,

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = -\frac{Cr^{1+p}\dot{\phi}^{1+p}}{m}. \tag{25}$$

An analytic solution is especially simple for $p = 0$ (drag force proportional to velocity).[4] As before, we use the derivative of the result (24) of the radial equation in the azimuthal equation (25) to find,

$$\dot{r} = -\frac{2C}{m}r, \tag{26}$$

and hence

$$r = r_0 e^{-2Ct/m}. \tag{27}$$

Thus, using eq. (24),

$$v_\phi = v_0 e^{Ct/m}. \tag{28}$$

In contrast, the radius (11) of the classical "atom" in an external magnetic field decreases as the field increases whether the azimuthal force applies a drag or a boost! Furthermore, when the azimuthal force on the classical "atom" is a drag the azimuthal velocity decreases in magnitude,[5] and increases when the force is a boost.

C Appendix: Adiabatic Invariance

A more formal approach to the problem of classical diamagnetism can be based on the (nonrelativistic) Lagrangian,[6]

$$\mathcal{L} = \frac{mv^2}{2} - e\frac{\mathbf{v}}{c}\cdot\mathbf{A} + eV = \frac{m(\dot{r}^2 + r^2\dot{\phi}^2)}{2} - \frac{er^2\dot{\phi}B(t)}{2c} + \frac{ee'}{r}, \tag{29}$$

[4]An analytic solution is also possible for the case of $p = -1$, *i.e.*, for a constant drag force [6].

[5]See sec. 2.1 of [9].

[6]In the present example the term $-e\mathbf{v}\cdot\mathbf{A}/c$ can be rewritten as $\boldsymbol{\mu}\cdot\mathbf{B}$, where $\boldsymbol{\mu}$ is the (orbital) magnetic moment.

for the electron (of charge $-e$) in the Coulomb potential $V = e'/r$ of the nucleus and the external magnetic field $\mathbf{B}(t) = B\,\hat{\mathbf{z}} = \mathbf{\nabla} \times \mathbf{A}$ where $\mathbf{A} = rB\,\hat{\boldsymbol{\phi}}/2$. As the Lagrangian (29) does not depend on coordinate ϕ, the canonical momentum p_ϕ is conserved, and we identify this as the z component of the canonical angular momentum,

$$l_z = p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi} - \frac{er^2 B}{2c} = L_z - \frac{er^2 B}{2c} = L_0 = mr_0 v_0\,, \tag{30}$$

where $L_z = mr^2\dot{\phi}$ is the mechanical angular momentum about the z axis. Thus,

$$\dot{\phi} = \frac{L_0^2}{mr^2} + \frac{eB}{2mc}\,, \tag{31}$$

where $eB(t)/2mc$ is the cyclotron (Larmor) frequency.

The radial equation of motion is,

$$m\ddot{r} = mr\dot{\phi}^2 - \frac{er\dot{\phi}B}{c} - \frac{ee'}{r^2} = \frac{L_0^2}{mr^3} - \frac{e^2 rB^2}{4mc^2} - \frac{L_0^2}{mr_0 r^2} \equiv -\frac{\partial U_{\rm eff}}{\partial r}\,, \tag{32}$$

where we note that $ee' = L_0^2/mr_0$ for the initial circular orbit. The effective radial potential $U_{\rm eff}$ is,

$$U_{\rm eff}(r,t) = \frac{L_0^2}{2mr^2} + \frac{e^2 r^2 B^2}{8mc^2} - \frac{L_0^2}{mr_0 r}\,. \tag{33}$$

However, the problem is not one of small oscillations about a fixed minimum of the effective potential. Rather, we suppose that the orbit at time t is nearly circular with radius $r(t)$ such that the effective potential is minimum at this time. That is, we set eq. (32) to zero, which leads again to eq. (10), to the solutions (11)-(13) and to the magnetic moment (18).

We have found one invariant in this problem, the canonical angular momentum l_z of eq. (30). Because this remains constant during changes in the magnetic field, whether these are adiabatic or not, we can certainly identify the canonical angular momentum as an adiabatic invariant. We could define the canonical magnetic moment μ_C as,

$$\mu_C = -\frac{e}{2mc} l_z = \mu_z + \frac{e^2 r^2 B^2}{2m^2c^2} \approx \mu_z + \frac{e^2 r_0^2 B^2}{2m^2c^2}\,, \tag{34}$$

which is also an adiabatic invariant. However, the ordinary magnetic moment μ_z is not an adiabatic invariant in this problem.

We can compare the present case of a classical "atom" to that of an electron in a uniform magnetic field. For bound motion in the latter case we cannot begin with zero magnetic field, but we suppose instead that $\mathbf{B} = B_0\,\hat{\mathbf{z}}$ at time $t = 0$.

Then, the Lagrangian is obtained from eq. (29) by setting $e' = 0$. The canonical angular momentum (30) is again (an adiabatic) invariant, while the radial equation of motion simplifies to $mr^2\dot{\phi} = er^2B/c$ for slow changes in the magnetic field B. Hence, the canonical angular momentum l_z can be written as,

$$l_z = mr^2\dot{\phi} - \frac{er^2 B}{2c} = \frac{er^2 B}{2c} = \frac{L_z}{2}\,, \tag{35}$$

and the canonical magnetic moment is,

$$\mu_C = -\frac{e}{2mc} l_z = \frac{\mu_z}{2}\,. \tag{36}$$

In this case the ordinary magnetic moment μ_z is also an adiabatic invariant. However, most discussions of the motion of charged particles in slowly varying magnetic fields do not emphasize that the ordinary magnetic moment is (somewhat accidentally) twice the invariant canonical magnetic moment, which may leave a misimpression as to the (non)invariance of the ordinary magnetic moment in other circumstances.

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