“Hidden” Momentum in a Charged, Rotating Cylinder

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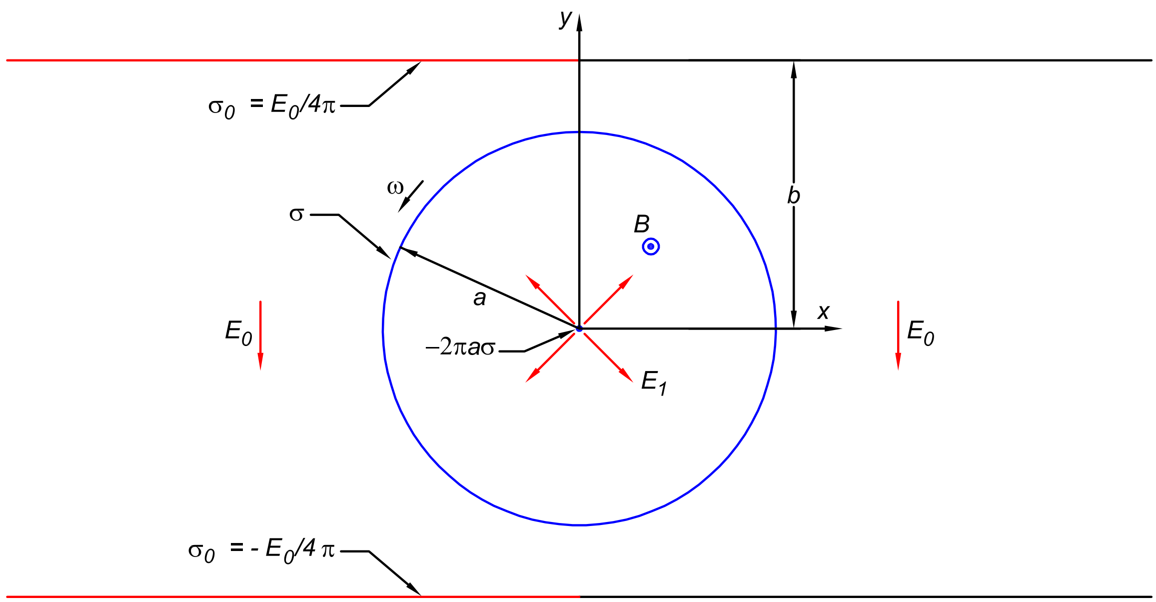
1 Problem

Discuss the motion of a uniformly charged, nonconducting cylinder, initially rotating about its axis, which cylinder lies between the plates of a capacitor, when the rotation drops zero.



The cylinder (of radius a) and the plates of the capacitor (in the planes $y = \pm b$) can move independently and without friction in the x-direction, where the z-axis is that of the cylinder. That axis can be electrically charged with the opposite sign to that of the cylinder such that the cylinder as a whole has zero charge. The (linear, isotropic) medium between the capacitor plates, including the interior of the cylinder, has relative permittivity ϵ and relative permeability μ. Ignore gravity.

This problem is a variant of that considered by Shockley [1], which is itself a variant of the Feynman disk paradox [2], which can be traced back to a discussion by Poincaré (1896) [3, 4], who built on a prescient argument of Darboux (1878) [5]. The geometry of this problem is also related to that of the Aharonov-Bohm effect [6].

2 Solution

2.1 Electric and Magnetic Fields

As the cylinder rotates with angular velocity ω it supports surface current $I = \omega a\sigma$ per unit length, where σ is the electrical charge density on the surface $r = a$ in cylindrical coordinates (r, ϕ, z). In the approximation of a very long cylinder, the magnetic field is uniform and axial inside it, and zero outside.

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = \mu \mathbf{H}, \qquad B_z = \frac{1}{r}\frac{\partial r A_\phi}{\partial r} = \mu H_z = \begin{cases} \frac{4\pi\mu I}{c} = \frac{4\pi\mu\omega a\sigma}{c} & (r < a), \\ 0 & (r > a), \end{cases} \tag{1}$$

in Gaussian units. The vector potential is azimuthal,

$$A_\phi = \frac{1}{2\pi r}\int B_z\, d\text{Area} = \begin{cases} \frac{rB_z}{2} = \frac{2\pi\mu\omega a\sigma r}{c} & (r < a), \\ \frac{a^2 B_z}{2r} = \frac{2\pi\mu\omega a^3\sigma}{cr} & (r > a). \end{cases} \tag{2}$$

When the angular velocity drops at rate $\dot\omega < 0$, the induced electric field is,

$$E_{\text{ind},\phi} = -\frac{1}{c}\frac{\partial A_\phi}{\partial t} = \begin{cases} -\frac{2\pi\mu\dot\omega a r\sigma}{c^2} & (r < a), \\ -\frac{2\pi\mu\dot\omega a^3\sigma}{c^2 r} & (r > a). \end{cases} \tag{3}$$

The capacitor plates are charged until the interior electric field is,

$$\mathbf{E}_0(|y| < b) = -E_0\,\hat{\mathbf{y}}, \qquad \mathbf{D}_0(|y| < b) = -\epsilon E_0\,\hat{\mathbf{y}}, \tag{4}$$

for which the surface-charge densities on the plates are,

$$\sigma_0(y = \pm b) = \pm\frac{D_0}{4\pi} = \pm\frac{\epsilon E_0}{4\pi}\,. \tag{5}$$

In addition, the electric field due to the charges on the cylinder and on its axis is,

$$\mathbf{E}_1(r < a) = \frac{\mathbf{D}_1(r < a)}{\epsilon} = \frac{2\pi a\sigma}{\epsilon r}\,\hat{\mathbf{r}}, \tag{6}$$

as the charge per unit length along the axis is $-2\pi a\sigma$. The electric field $\mathbf{E}_0$ is zero outside the capacitor, and $\mathbf{E}_1$ is zero outside the cylinder in the 2-dimensional approximation.

2.2 Motion of the Capacitor

As the angular velocity drops to zero, $\dot\omega < 0$, the induced electric field (3) acts on the positively charged plate at $y = b$ to push it in the $-x$ direction (and also pushes the plate towards $y = 0$, but we suppose the plates are held apart by the medium between them). Likewise, the electric field induced on the negatively charged plate at $y = -b$ pushes it in the $-x$ direction, so the capacitor will move in this direction (assuming that there is no friction between the plates and the medium between).

The sum of the force per unit length in y on the two plates is,

$$F_{\text{capacitor},x} = 2\int_{-\infty}^{\infty} \sigma_0 E_{\text{ind},x}(y = b)\, dx = -2\int_{-\infty}^{\infty} \frac{\epsilon E_0}{4\pi}\frac{2\pi\mu\dot\omega a^3\sigma b}{c(x^2 + b^2)}\, dx = \frac{\pi\epsilon\mu E_0\dot\omega a^3\sigma}{c}\,. \tag{7}$$

so the capacitor (considered to be separate from the medium between its plates) takes on mechanical momentum per unit length,

$$P_{\text{capacitor},x} = \int F_{\text{capacitor},x}\, dt = -\frac{\pi\epsilon\mu E_0\omega a^3\sigma}{c^2}\,, \tag{8}$$

when the angular velocity drops from ω to zero. If the capacitor is free to move relative to the medium/cylinder, it has final velocity in the $-x$ direction.

2.3 Motion of the Cylinder

The total force of the electric field $\mathbf{E}_0$ on the charged cylinder and axis is zero, so this force produces no motion of the cylinder, presuming that the cylinder is held fixed with respect to its axis, while being free to rotate about the latter (with no friction against the media on either side of the cylindrical surface).

As the angular velocity drops to zero, $\dot{\omega} < 0$, the electric field induced at the cylinder $r = a$ opposes the reduction in ω (Lenz' law) but does not prevent it from dropping to zero. However, the sum of the forces of the induced field on the cylinder is also zero, and leads to no net motion of its center of mass.[1]

2.4 Motion of the Medium

We suppose that the medium between the capacitor plates moves with the axis of the cylinder, and *vice versa*. However, this medium does not rotate along with the cylinder (although in principle the cylindrical portion of the medium inside the charged cylinder could be rotating along with it). This medium supports bound surface charges densities at on its surfaces at $y = \pm b$ (as well as on its surfaces next to the cylinder and axis; but these densities do not lead to net forces),

$$\sigma_b(y = \pm b) = \mathbf{P} \cdot \hat{\mathbf{n}} = \frac{(\epsilon - 1)\mathbf{E}_0}{4\pi} \cdot (\mp \hat{\mathbf{y}}) = -\frac{\epsilon - 1}{\epsilon} \sigma_0(y = \pm b) = \mp \frac{(\epsilon - 1)E_0}{4\pi} . \tag{9}$$

The sum of the force per unit length in y on the two surfaces of the medium next to the capacitor plates is,

$$F_{\mathrm{medium},x} = 2 \int_{-\infty}^{\infty} \sigma_b E_{\mathrm{ind},x}(y = b)\, dx = 2 \int_{-\infty}^{\infty} \frac{(\epsilon - 1)E_0}{4\pi} \frac{2\pi\mu\dot{\omega}a^3\sigma b}{c(x^2 + b^2)}\, dx = \frac{\pi\epsilon\mu E_0 \dot{\omega} a^3 \sigma}{c} . \tag{10}$$

so the medium (considered to be separate from the capacitor) takes on mechanical momentum per unit length,

$$P_{\mathrm{medium},x} = \int F_{\mathrm{medium},x}\, dt = \frac{\pi(\epsilon - 1)\mu E_0 \omega a^3 \sigma}{c^2} , \tag{11}$$

when the angular velocity drops from ω to zero. If the medium is free to move relative to the capacitor (carrying the rotating, charged cylinder along with it), the final velocity of the medium+cylinder is in the $+x$ direction.

Meanwhile the capacitor moves in the $-x$ direction (sec. 2.2) as the angular velocity of the cylinder falls to zero. The final (mechanical) momentum per unit length of the entire system is,

$$P_{\mathrm{total},x} = P_{\mathrm{medium},x} + P_{\mathrm{capcitor},x} = -\frac{\pi\mu E_0 \omega a^3 \sigma}{c^2} , \tag{12}$$

This appears to be a violation of conservation of momentum, as noted by Shockley [1] for a closely related example. Following him, we now look more carefully for "hidden" forces, energy and momenta in the system.

[1] If the cylinder were to move, it would carry along the medium inside and outside it.

2.5 Flow of Energy in Electromagnetic Fields

A major addition to Maxwell's electrodynamics was made in 1884 by Poynting [7] and Heaviside [8] when they provided a description of the flow of energy in electromagnetic fields.

Poynting's theorem expresses energy conservation in electromagnetic phenomena in the form,[2]

$$\frac{\partial \text{ energy density}}{\partial t} + \boldsymbol{\nabla} \cdot \textbf{energy current density} = \text{source power density}. \tag{13}$$

In the standard version, the sources on the right side are "nonelectromagnetic" in character, such as batteries or dynamos that convert chemical (*i.e.*, quantum electrodynamic) or "mechanical" (another form of quantum field) energy into "electromagnetic" form as understood in the context of "classical" electrodynamics.

We find it instructive to characterize the nonelectromagnetic power source by a nonelectromagnetic field $\mathbf{E}'$ that acts on the "free" conduction current $\mathbf{J}_{\rm free}$ according to an extension of Ohm's law,

$$\mathbf{J}_{\rm free} = \sigma(\mathbf{E} + \mathbf{E}'), \tag{14}$$

where $\mathbf{E}$ is the usual electric field and σ is the conductivity of the medium that supports the conduction current. This permits us to relate the nonelectromagnetic field $\mathbf{E}'$ to electromagnetic quantities,

$$\mathbf{E}' = \frac{\mathbf{J}_{\rm free}}{\sigma} - \mathbf{E}. \tag{15}$$

The total density of power delivered by the nonelectromagnetic source to the electromagnetic system is then,

$$P_{\rm nonelectromagnetic,total} = \mathbf{J}_{\rm free} \cdot \mathbf{E}' = \frac{J_{\rm free}^2}{\sigma} - \mathbf{J}_{\rm free} \cdot \mathbf{E}. \tag{16}$$

The first term on the right side of eq. (16) is the density of Joule heating of the conductive medium, and we regard this power as "lost" with respect to the electromagnetic system. In contrast, the second term represents power that is transferred from the nonelectromagnetic system into energy stored, or flowing, within the electromagnetic system,

$$\begin{aligned} P_{\rm nonelectromagnetic,transferred} = -\mathbf{J}_{\rm free} \cdot \mathbf{E} &= -\frac{c}{4\pi}\mathbf{E} \cdot \left(\boldsymbol{\nabla} \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}\right) \qquad (17) \\ &= \frac{c}{4\pi}\left(\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}) - \mathbf{H} \cdot \boldsymbol{\nabla} \times \mathbf{E} + \mathbf{E} \cdot \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}\right) \\ &= \frac{c}{4\pi}\left(\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}) + \mathbf{E} \cdot \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\right), \end{aligned}$$

[2] Apparently, the form (13) was first considered by Umov [9] as an extrapolation to energy flow of Euler's continuity equation for mass flow [10].

where we have used the third and fourth macroscopic Maxwell equations and a vector-calculus identity. Of course, $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ and $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$, where $\mathbf{P}$ and $\mathbf{M}$ are the densities of electric and magnetic dipoles. We identify the Poynting vector,

$$\mathbf{S}^{(\text{Poynting})} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B} - c\mathbf{E} \times \mathbf{M}, \tag{18}$$

as describing the flow (current density) of electromagnetic energy, and,

$$\frac{\partial u^{(\text{Poynting})}}{\partial t} = \frac{1}{4\pi}\left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}\right) = \frac{E^2 + B^2}{8\pi} + \mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} - \mathbf{M} \cdot \frac{\partial \mathbf{B}}{\partial t}\,. \tag{19}$$

as the time rate of change of the electromagnetic energy density u.

While Poynting's theorem (17) clearly suggests that $\mathbf{S}^{(\text{Poynting})}$ describes the flow of electromagnetic energy, it does not definitively identify the electromagnetic energy density u. Further, the expressions (18)-(19) include $\mathbf{P}$ and $\mathbf{M}$, which are not electromagnetic fields so much as fields related to charge and current distributions. This leaves open the possibility that some alternative form of eq. (17) might be preferable.

2.5.1 Poynting's Theorem for Linear Media

For so-called linear media in which $\mathbf{E}$ is proportional to $\mathbf{D}$ and $\mathbf{B}$ is proportional to $\mathbf{H}$, the expression (19) is a perfect differential and we can write the electromagnetic energy density $u^{(\text{Poynting})}$ as,

$$u^{(\text{Poynting})} = \frac{\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}}{8\pi} = \frac{E^2 + B^2}{8\pi} + \frac{\mathbf{E} \cdot \mathbf{P}}{2} - \frac{\mathbf{B} \cdot \mathbf{M}}{2}\,, \tag{20}$$

and Poynting's theorem reads,

$$\boldsymbol{\nabla} \cdot \mathbf{S}^{(\text{Poynting})} + \frac{\partial u^{(\text{Poynting})}}{\partial t} = -\frac{4\pi}{c}\mathbf{J}_{\text{free}} \cdot \mathbf{E} = P_{\text{nonelectromagnetic,transferred}}\,. \tag{21}$$

2.5.2 An Alternative

The standard version (21) of Poynting's theorem assumes that only free currents can be sources or sinks of electromagnetic field energy. But as we examine the possibility of "hidden" forces, energy and momentum it is useful to consider at least one alternative, that the "total" current can emit/absorb field energy,[3]

$$\mathbf{J}_{\text{total}} = \mathbf{J}_{\text{free}} + \frac{\partial \mathbf{P}}{\partial t} + c\boldsymbol{\nabla} \times \mathbf{M}, \tag{22}$$

[3]We do not consider Maxwell's displacement current, $(1/4\pi)\partial\mathbf{D}/\partial t$, in the current $\mathbf{J}$ that couples to $\mathbf{E}$ as a source or sink of field energy, although logically we could. Indeed, following this line of thought leads to 729 variants of Poynting's theorem, most of which are rather implausible physically as they imply that the electromagnetic fields are sources of themselves [11]. Of course, such behavior occurs in gravity and the strong interaction, which are described by nonlinear field theories.

where the second and third terms in eq. (22) and the so-called (bound) polarization and magnetization currents. This assumptions leads to a variant of Poynting's theorem,

$$\nabla \cdot \mathbf{S}^{(\mathrm{E-B})} + \frac{\partial u^{(\mathrm{E-B})}}{\partial t} = P_{\mathrm{source}} = -\frac{4\pi}{c}\mathbf{E} \cdot \mathbf{J}_{\mathrm{total}},$$
$$\mathbf{S}^{(\mathrm{E-B})} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B}, \qquad u^{(\mathrm{E-B})} = \frac{E^2 + B^2}{8\pi}\,. \tag{23}$$

This variant is the macroscopic average of the microscopic version of Poynting's theorem, where here the **E** and **B** are macroscopic averages over their microscopic counterparts. Equation (23) might be called the "pure" electromagnetic field version of Poynting's theorem in that for all other variants the "material" fields **P** or **M** appear somewhere in **S** or u.[4]

2.6 Electromagnetic Field Momentum

The notion of momentum stored in an electromagnetic field was first developed by J.J. Thomson, building on Poynting's description of the flow of energy in the field.

2.6.1 Radiation Pressure and the Momentum of Light

Apparently, Kepler considered the pointing of comets' tails away from the Sun as evidence for radiation pressure of light [17]. After his unification of electricity, magnetism and light [22], Maxwell argued (sec. 792 of [23]) that the radiation pressure P of light is equal to its energy density u,

$$P = u = \frac{D^2}{4\pi} = \frac{H^2}{4\pi} \tag{26}$$

for an electromagnetic wave with fields **D** and **H** in vacuum, but he did not explicitly associate this pressure with momentum in the electromagnetic field.[5]

[4]The "alternative" argument of this section was presented by Fano, Chu and Adler (1960) as the standard argument in sec. 7.10 of [12]. However, those authors were immediately disconcerted by the implication (their sec. 5.4) that a (nonconducting) permanent magnet in an electric field supports a flow of energy from one part of the magnet to another. This flow of energy is essential to the discussion in a companion note [13] to the present one.

The discomfort of Fano, Chu and Adler with eq. (23) led to the suggestion of another alternative,

$$\nabla \cdot \mathbf{S}^{(\mathrm{Poynting})} + \frac{\partial u^{(\mathrm{E-B})}}{\partial t} = -\frac{4\pi}{c}\mathbf{E} \cdot \left(\mathbf{J}_{\mathrm{free}} + \frac{\partial \mathbf{P}}{\partial t}\right) + 4\pi\mathbf{M} \cdot \frac{\partial \mathbf{B}}{\partial t} \tag{24}$$

in eq. (4) of [14] and in eq. (7.70) of [15], which was rewritten in eq. (5) of[16] as,

$$\nabla \cdot \mathbf{S}^{(\mathrm{Poynting})} + \frac{\partial u^{(\mathrm{E-B})}}{\partial t} = -\frac{4\pi}{c}\mathbf{E} \cdot \left(\mathbf{J}_{\mathrm{free}} + \frac{\partial \mathbf{P}}{\partial t}\right) + 4\pi\left(\frac{\partial(\mathbf{B}\cdot\mathbf{M})}{\partial t} - \mathbf{B} \cdot \frac{\partial \mathbf{M}}{\partial t}\right). \tag{25}$$

[5]Maxwell (and Thomson and Lorentz and most others influenced by the concept of a material aether), regarded the fields **D** and **H** as more "basic" than **E** and **B**.

Building on Faraday's electrotonic state,[6] Maxwell did have a conception of electromagnetic momentum, computed as [22, 23],

$$\mathbf{P}_{\rm EM}^{\rm (Maxwell)} = \int \frac{\rho \mathbf{A}^{(C)}}{c}\, d{\rm Vol}, \tag{27}$$

where ρ is the electric charge density and $\mathbf{A}^{(C)}$ is the vector potential in the Coulomb gauge (that Maxwell used prior to the explicit recognition of gauge conditions [24]), but the form (3) seems to associate the momentum with charges rather than with fields.

In 1891 [25], Thomson noted that a sheet of electric displacement $\mathbf{D}$ (parallel to the surface) which moves perpendicular to its surface with velocity $\mathbf{v}$ must be accompanied by a sheet of magnetic field $\mathbf{H} = \mathbf{v}/c \times \mathbf{D}$ according to the free-space Maxwell equation $\boldsymbol{\nabla} \times \mathbf{H} = (1/c)\, \partial \mathbf{D}/\partial t$.[7] Then, the motion of the energy density of these sheets implies there is also a momentum density, eqs. (2) and (6) of [25],

$$\mathbf{p}_{\rm EM}^{\rm (Thomson)} = \frac{\mathbf{D} \times \mathbf{H}}{4\pi c}\, . \tag{28}$$

In 1893, Thomson transcribed much of his 1891 paper into the beginning of *Recent Researches* [28], adding the remark (p. 9) that the momentum density (4) is closely related to the Poynting vector [7, 29],[8,9] now commonly written as eq. (18),

$$\mathbf{S}^{\rm (Poynting)} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H}. \tag{29}$$

The form (28) was also used by Poincaré in 1900 [33], following Lorentz' convention [34] that the force on electric charge q be written $q(\mathbf{D} + \mathbf{v}/c \times \mathbf{H})$ and that the Poynting vector is $(c/4\pi)\, \mathbf{D} \times \mathbf{H}$. In 1903 Abraham [35] argued for,

$$\mathbf{p}_{\rm EM}^{\rm (Abraham)} = \frac{\mathbf{E} \times \mathbf{H}}{4\pi c} = \frac{\mathbf{S}^{\rm (Poynting)}}{c^2}\, , \tag{30}$$

and in 1908 Minkowski [36] advocated the form,[10,11]

$$\mathbf{p}_{\rm EM}^{\rm (Minkowski)} = \frac{\mathbf{D} \times \mathbf{B}}{4\pi c}\, . \tag{31}$$

[6] Faraday first speculated on an *electro-tonic* state in Art. 60 of [18]. Other mentions by Faraday of the electrotonic state include Art. 1661 of [19], Arts. 1729 and 1733 of [20], and Art. 3269 of [21].

[7] Variants of this argument were given by Heaviside in 1891, sec. 45 of [26], and much later in sec. 18-4 of [2], where it is noted that Faraday's law, $\boldsymbol{\nabla} \times \mathbf{E} = -(1/c)\, \partial \mathbf{B}/\partial t$, combined with the Maxwell equation for $\mathbf{H}$ implies that $v = c$ in vacuum, which point seems to have been initially overlooked by Thomson, although noted in sec. 265 of [27].

[8] The idea that an energy flux vector is the product of energy density and energy flow velocity seems to be due to Umov [9], based on Euler's continuity equation [10] for mass flow, $\boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = -\partial \rho/\partial t$.

[9] Thomson argued, in effect, that the field momentum density (4) is related by $\mathbf{p}_{\rm EM} = \mathbf{S}/c^2 = u\mathbf{v}/c^2$ [25, 28]. See also eq. (19), p. 79 of [26], and p. 6 of [30]. It turns out that the energy flow velocity defined by $\mathbf{v} = \mathbf{S}/u$ can exceed c (see, for example, sec. 2.1.4 of [31] and sec. 4.3 of [32].

[10] Minkowski, like Poynting [7], Heaviside [29] and Abraham [35], wrote the Poynting vector as $\mathbf{E} \times \mathbf{H}$. See eq. (75) of [36].

[11] For some remarks on the "perpetual" Abraham-Minkowski debate see [37].

Thomson did not relate the momentum density (11) to the radiation pressure of light, eq. (9), until 1904 (p. 355 of [38]; see also [39]) when he noted that $P = F/A = c\, p_{\rm EM} = D^2/4\pi = H^2/4\pi$ for fields moving with speed c in vacuum, for which $D = H$. He also gave an argument (p. 348 of [38]) that the forms (3) and (4) for field momentum are equivalent once the sources of the fields are taken into account.[12]

Among the various forms for the electromagnetic field momentum, that associated with the "E-B" Poynting vector of eq. (23) is the field-momentum density,

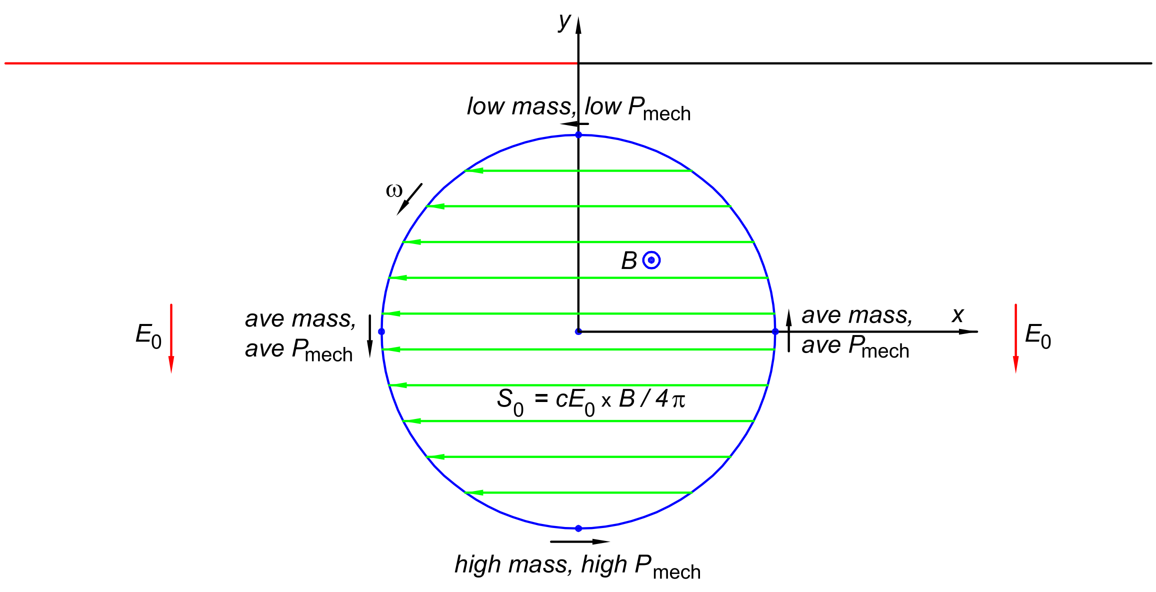
$$\mathbf{p}_{\rm EM}^{\rm (E-B)} = \frac{\mathbf{S}^{\rm (E-B)}}{c^2} = \frac{\mathbf{E}\times\mathbf{B}}{4\pi c} \qquad (32)$$

2.6.2 Field Momentum inside the Rotating Cylinder

In the present example the magnetic field is nonzero only inside the cylinder, so the field-only momentum per unit length along z is,

$$\mathbf{P}_{\rm EM}^{\rm (E-B)} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol}' = \int \frac{\mathbf{E}_0\times\mathbf{B}}{4\pi c}\, d{\rm Vol}' + \int \frac{\mathbf{E}_1\times\mathbf{B}}{4\pi c}\, d{\rm Vol}' = -\frac{\pi\mu a^3 E_0\omega\sigma}{c^2}\,\hat{\mathbf{x}}, \qquad (33)$$

which is only due to the field $\mathbf{E}_0$ of the capacitor, with no net contribution from the radial electric field $\mathbf{E}_1$ inside the cylinder.



This initial field momentum (per unit length) is identical to the final mechanical momentum (12) of the system after the magnetic field, and the field energy, have dropped to zero. Hence, it is suggestive (but not definitive) that the final mechanical momentum of the system is the result of conversion of field momentum into mechanical momentum as the field vanishes.

But, this leaves open the issue that it seems contradictory for a system initially "at rest" to end up with nonzero total momentum.

[12]Possibly, Thomson delayed publishing the relation of radiation pressure to his expression (4) until he could demonstrate its equivalence to Maxwell's form (10). For other demonstrations of this equivalence, see Appendix B of [40], and [41].

For completeness, we note that the Abraham and Minkowski momenta are,

$$\mathbf{P}_{\rm EM}^{\rm (A)} = \int \frac{\mathbf{E}_0 \times \mathbf{H}}{4\pi c}\, d{\rm Vol}' = -\frac{\pi a^3 E_0 \omega\sigma}{c^2}\,\hat{\mathbf{x}}, \qquad \mathbf{P}_{\rm EM}^{\rm (M)} = \int \frac{\mathbf{D}_0 \times \mathbf{B}}{4\pi c}\, d{\rm Vol}' = -\frac{\pi \epsilon\mu a^3 E_0 \omega\sigma}{c^2}\,\hat{\mathbf{x}}. \quad (34)$$

2.7 "Hidden" Mechanical Momentum

2.7.1 A Suggestive Argument

A related example, in which momentum appeared not to be conserved in an electromechanical system initially "at rest", led Shockley in 1967 to develop the notion of "hidden" mechanical momentum [1], *i.e.*, that the total mechanical momentum of a system is not necessarily the product of its mechanical mass/energy and the velocity of its center of mechanical mass/energy.[13,14] This notion was clarified in important ways by Coleman and Van Vleck [43].

Here, we note with Poincaré [33] and Abraham [35] that the momentum density $\mathbf{p}_{\rm EM}$ equals the Poynting vector divided by c^2, $\mathbf{p}_{\rm EM} = \mathbf{S}/c^2$. Then, the figure above indicates that energy is transferred from the currents on the right to those on the left. According to Einstein's $U = mc^2$, such transfer of energy should change the masses of the charges in the currents, and hence also there momenta change. Qualitatively, the lowest mass/momentum is for the currents at $(x, y) = (0, a)$, and the highest at $(0, -a)$. For counterclockwise rotation, the differing momenta of the currents as a function of azimuth lead to a net momentum in the $+x$ direction. This "hidden" mechanical momentum[15] should be equal and opposite to the initial field momentum, such that the total initial momentum of the system is zero.

Then, as the rotation of the cylinder vanishes, this "hidden" mechanical momentum should be converted into "overt" momentum of the cylinder/medium, which should end up with momentum opposite to that of the capacitor plate.

2.7.2 Detailed Computation for the Rotating Cylinder

The flux of electromagnetic energy into a surface is given by the normal component of the Poynting vector. As seen earlier, only the Poynting vector associated with the electric field $\mathbf{E}_0$ of the capacitor is relevant for the momentum of the system, so the Poynting flux into/out of the rotating cylinder is, for the E-B form (23) of for the Poynting vector,

$$\begin{aligned}\frac{dU}{dt\, d{\rm Area}}(r=a) &= \hat{\mathbf{r}} \cdot \mathbf{S}^{(\rm E-B)} = \frac{c}{4\pi}\hat{\mathbf{r}} \cdot \mathbf{E}_0 \times \mathbf{B}(r=a) = \frac{c}{4\pi}\hat{\mathbf{r}} \times \mathbf{E}_0 \cdot \mathbf{B}(r=a) \\ &= \frac{c}{4\pi} E_{0,\phi} B_z(r=a) = \frac{c}{4\pi}(-E_0 \cos\phi)\frac{4\pi\omega a\sigma}{c} = -E_0 \omega a \sigma \cos\phi, \quad (35)\end{aligned}$$

[13]The first such example was given in 1904 by J.J. Thomson on p. 348 of [38]. See also [39]. For examples with "hidden" mechanical momentum in systems with an electric dipole in a magnetic field due to current loops, all "at rest", such that various equal and opposite "overt" mechanical momenta arise as the electromagnetic fields are brought to zero in various ways, see [42], especially secs. IV and V.

[14]For a general discussion on the meaning of "hidden" momentum see [40].

[15]We call this momentum "mechanical", but in the view of quantum field theory, all "mechanical mass, energy and momentum is field mass, energy, momentum. It is tempting to speculate that this "hidden mechanical" momentum is electromagnetic in origin, but the premise that all mass is electromagnetic mass (first postulated by J.J. Thomson is 1881 [44]) has not stood the test of time. A more contemporary speculation is that "hidden mechanical" momentum is as aspect of the Higgs field.

The mass of the material that absorbs or emits this Poynting flux changes at the rate,

$$\frac{dm(r=a)}{dt\,d\text{Area}} = \frac{1}{c^2}\frac{dU}{dt\,d\text{Area}} = -\frac{E_0\omega a\sigma\cos\phi}{c^2}\,. \tag{36}$$

On the other hand, the present system is "at rest", and we expect that the mass at a given location (r, ϕ, z) cannot depend on time.

Of course, the cylinder is rotating, which transports the changed masses azimuthally, and permits a steady-state mass-density distribution per unit area in the lab frame,

$$\sigma_{\text{mass}}(r=a,\phi) = \sigma_{\text{mass},0}(r=a) + \Delta\sigma_{\text{mass}}(r=a,\phi). \tag{37}$$

During small time interval dt the mass per unit length along z in an element $a\,d\phi$ of the rotating cylinder at $r = a$ changes by amount,

$$dm = \frac{dm(r=a)}{dt\,d\text{Area}}\,dt\,a\,d\phi + ad\phi(\Delta\sigma_{\text{mass}}(\phi-\omega dt) - \Delta\sigma_{\text{mass}}(\phi)), \tag{38}$$

but which change must be zero for a system "at rest". Hence,

$$0 = -\frac{E_0\omega a\sigma\cos\phi}{2\pi c^2} - \omega\frac{d\Delta\sigma_{\text{mass}}}{d\phi}\,, \qquad \Delta\sigma_{\text{mass}}(r=a)\phi) = -\frac{E_0 a\sigma\sin\phi}{c^2}\,. \tag{39}$$

The corresponding density of azimuthal momentum per unit length along z in the rotating cylinder at $r = a$ is,

$$\frac{dp_\phi(r=a)}{d\phi} = a\sigma_{\text{mass}}(a\omega) = a^2\omega\sigma_{\text{mass},0}(r=a) - \frac{E_0 a^3\omega\sigma\sin\phi}{c^2}\,. \tag{40}$$

Consequently, the (azimuthal) momentum of the rotating cylinder varies with ϕ, and the cylinder has nonzero total momentum, even though it is part of a system "at rest",

$$P_x(r=a) = \int_0^{2\pi}\frac{dp_\phi(r=a)}{d\phi}(-\sin\phi)\,d\phi = \frac{\pi E_0 a^3\omega\sigma}{c^2}\,. \tag{41}$$

This possibly surprising, nonzero momentum of the rotating cylinder has been called its "hidden" mechanical momentum.

Thus, in the case that the medium has permeability $\mu = 1$, the momentum (41) of the rotating cylinder is equal and opposite to the momentum (33) stored in the **E** and **B** fields. The total momentum of the system is zero when it is "at rest", but with the charged cylinder in rotation.

If the rotation ceases, the momentum (41) drops to zero, resulting in a force on the cylinder that converts the "hidden" mechanical momentum of the rotating cylinder (whose center does not move) into "overt" momentum of the cylinder (whose center then moves in the $+x$ direction).

In the final state, the capacitor and the cylinder (+ medium with $\mu = 1$) are moving in opposite directions, with equal and opposite momenta.

This resolves the "paradox" noted at the end of sec. 2.3, at least for the case that $\mu = 1$ in the medium.

We note that if we had used the Poynting vector (29) for computation of the transfer of mass/energy from one side of the rotating cylinder to the other, and used the Abraham field-momentum density (30), which leads to the field momentum given in eq. (34), then the "paradox" is again resolved. Further, the "hidden" mechanical momentum of the rotating cylinder is still given by eq. (41) even when the medium has $\mu \neq 1$.

However, the paradox would not be resolved if the Minkowski field-momentum density (31) were used.

2.7.3 Detailed Computation for the Cylindrical Medium

If the medium has permeability μ different from unity, the "hidden" mechanical momentum of the rotating cylinder is not equal in magnitude to the momentum (33) stored in the field, although their directions are opposite. Recall that eq. (33) holds for the assumption that the field-momentum density is the E-B form (32), in which case the bound currents in the medium also serve as sources/sinks for the Poynting flux. Hence, we need to compute the "hidden" mechanical momentum in the bound surface currents.[16]

The magnetic field B_z at the surface $r = a^-$ of the inner cylinder of the medium is $2\mu\omega a\sigma/c$, so the Poynting flux into/out of this surface is, for the E-B form (23) of for the Poynting vector,

$$\frac{dU}{dt\, d\text{Area}}(r = a^-) \;=\; \frac{c}{4\pi}\hat{\mathbf{r}} \cdot \mathbf{E}_0 \times \mathbf{B}(r = a^-) = -\mu E_0 \omega a\sigma \cos\phi. \tag{42}$$

The mass of the this surface of the cylindrical medium thereby changes at the rate,

$$\frac{dm(r = a^-)}{dt\, d\text{Area}} = \frac{1}{c^2}\frac{dU}{dt\, d\text{Area}} = -\frac{\mu E_0 \omega a\sigma \cos\phi}{c^2}. \tag{43}$$

The cylindrical medium is "at rest", so its mass does not vary with time, but rather mass is convected around the surface such that the surface mass density is constant in time, but varies with azimuth ϕ.

The outer surface of the cylinder of the medium supports a bound surface current density,

$$\mathbf{K}_{\text{bound}}(r = a^-) = c\mathbf{M} \times \hat{\mathbf{n}} = \frac{c(\mu - 1)\mathbf{H}}{4\pi} \times \hat{\mathbf{r}} = \frac{(\mu - 1)\omega a\sigma}{2}\,\hat{\boldsymbol{\phi}}, \tag{44}$$

This surface current density is not due to an actual electric surface charge density σ that rotates with angular velocity ω, but it is equivalent to it.

For a steady mass distribution, as required for the system to be "at rest", an argument equivalent to that in the preceding section indicates that the part of the mass density on the surface of the cylindrical medium which varies with ϕ is,

$$\Delta\sigma_{\text{mass}}(r = a^-, \phi) = -\frac{(\mu - 1)E_0 a\sigma \sin\phi}{c^2}. \tag{45}$$

[16] If the Poynting vector (29) and the Abraham field-momentum density (30) were used, the bound surface currents would not absorb/emit field energy of the form (20). In this case, the present section is irrelevant.

This mass distribution is effectively convected with angular velocity ω, such that the corresponding density of azimuthal momentum per unit length along z on the surface at $r = a^-$ of the cylindrical medium at $r = a^-$ is,

$$\frac{dp_\phi(r = a^-)}{d\phi} = a\Delta\sigma_{\rm mass}(a\omega) = -\frac{(\mu - 1)E_0 a^3 \omega\sigma \sin\phi}{c^2}\,, \tag{46}$$

since only the mass difference $\Delta\sigma_{\rm mass}$ is effectively convected. Consequently, the (azimuthal) momentum of the cylindrical medium varies with ϕ, and this cylinder has nonzero total momentum, even though it "at rest",

$$P_x(r = a^-) = \int_0^{2\pi} \frac{dp_\phi(r = a^-)}{d\phi}(-\sin\phi)\, d\phi = \frac{\pi(\mu - 1)E_0 a^3 \omega\sigma}{c^2}\,. \tag{47}$$

Hence, the total mechanical momentum of the medium + rotating cylinder is,

$$P_{{\rm cylinder+medium},x} = P_x(r = a^-) + P_x(r = a) = \frac{\pi\mu E_0 a^3 \omega\sigma}{c^2}\,. \tag{48}$$

This is equal and opposite to the E-B field momentum (33) when the medium has permeability $\mu \neq 1$.

Thus, the "paradox" of the end of sec. 2.3 is resolved for any value of the permeability, either using the E-B Poynting vector (28) and field-momentum density (32), or by use of the Poynting vector (29) and the Abraham field-momentum density (30).

2.8 Comments

This example, like that of Shockley [1], illustrates that consistency of the laws of mechanics in electromechanical systems "at rest" implies that the electrical currents can contain a nonzero net momentum, called "hidden" momentum by Shockley. The transport of mass/energy from one part of the system to another by the Poynting vector indicates the need for this "hidden" momentum, and permits computation of this tiny quantity in the present example.

However, there are at least two reasonable assumptions as to the form of the Poynting vector in macroscopic electrodynamics, eqs. (23) and (29), and these two assumptions lead to differing values of the nonzero, equal and opposite, electromagnetic field momentum and "hidden" mechanical momentum for the system "at rest". The present example does not indicate that one or the other of these two assumptions is more "correct".

A different example, in which a uniformly, azimuthally magnetized toroid is between the electrodes of a cylindrical capacitor [13], has $\mathbf{H} = 0$ everywhere but B_ϕ nonzero inside the toroid, and appears to favor use of the E-B Poynting vector (23) and the associated field-momentum density (32).

Consideration of these two examples indicate that both the Abraham momentum density (30) and the Minkowski momentum density (31) are disfavored in accounting for observable final motions of parts of systems initially "at rest".

2.8.1 "Hidden" Momentum via Work rather than the Poynting Vector

Many explanations of "hidden" mechanical momentum proceed via consideration of the work done by the electric field on the moving charges that comprise the electrical current [42, 15, 45, 46, 47].[17] This is appealing from the point of view of mechanics, but leaves open the question of where does the energy gained by the charges come from?

A tacit view associated with the work done by the electric field is that the energy gained by the charges has come from the (distant) sources of the electric field. However, this view is not very "Maxwellian", with its implication of action at a distance, or that the energy flowed along lines of the electric field that does the work. In contrast, the view considered here is that the Poynting vector describes the flow of electromagnetic energy, which indicates that the flow of energy is perpendicular to lines of the electric (and magnetic) field. Indeed, in the present example, the energy gained by charges on the rotating disk comes from energy lost by other charged particles elsewhere on the same disk, rather than from the charges on the capacitor plates (or from the electric field of the capacitor).

That is, the arguments presented here have the possible appeal of being more "Maxwellian" than those based on work done by the electric field, although both approaches succeed in accounting for the "hidden" mechanical momentum of the electrical currents.[18]

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[18] Penfield and Haus [15] were the first to deduce the existence of "hidden" mechanical momentum from consideration of the work done on electrical currents by an external electric field. However, they also supposed that the Poynting's theorem had the form (18) such that the energy gained by the currents could not have been transmitted through the electromagnetic field if those currents were of the Ampèrian form $c\boldsymbol{\nabla} \times \mathbf{M}$ associated with magnetization $\mathbf{M}$.

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