# A Rolling Cylinder Problem

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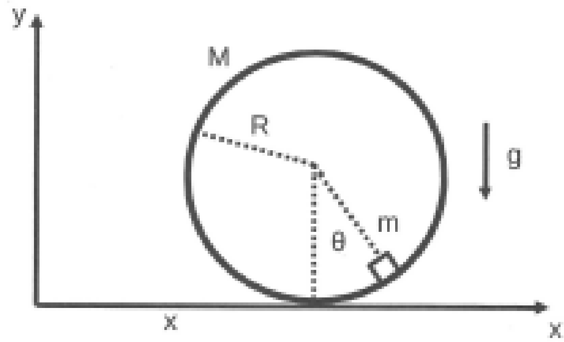
A hollow cylinder of mass $M$ and radius $R$ lies, initially at on a frictionless horizontal surface with its axis in the $y$ direction.[1,2] A point mass $m$ is fixed to the surface of the cylinder, and the radius from the center of the cylinder to mass $m$ makes angle $\theta$ to the vertical. The system is initially at rest, with mass $m$ at angle $\theta_0$.




There is no external horizontal force on the system (of masses $M$ and $m$), so its center of mass remains at rest (and its total horizontal momentum is always zero).

We define $x = 0$ to be the $x$-coordinate of the center of mass of this system,

$$(M + m)x_{\rm cm} = 0 = Mx_{\rm cyl} + mx_m = Mx_{\rm cyl} + m(x_{\rm cyl} + R\sin\theta), \qquad (1)$$

$$x_{\rm cyl} = -\frac{mR\sin\theta}{M+m}, \qquad \dot{x}_{\rm cyl} = -\frac{mR\cos\theta\,\dot\theta}{M+m}. \qquad (2)$$

In general, the cylinder has horizontal velocity $\dot{x}_{\rm cyl}$ and rotates with angular velocity $\dot\theta$. Hence, the kinetic energy of the (hollow) cylinder is,

$$T_{\rm cyl} = \frac{M\dot{x}_{\rm cyl}^2}{2} + \frac{MR^2\dot\theta^2}{2} = \frac{MR^2\,\dot\theta^2}{2}\left(1 + \frac{m^2\cos^2\theta}{(M+m)^2}\right). \qquad (3)$$

The coordinates of mass $m$ are, using eq. (3),

$$x_m = x_{\rm cyl} + R\sin\theta = R\sin\theta\left(1 - \frac{m}{M+m}\right) = \frac{M}{M+m}R\sin\theta, \qquad y_m = R(1-\cos\theta), \qquad (4)$$

$$\dot{x}_m = \frac{M}{M+m}R\cos\theta\,\dot\theta, \qquad \dot{y}_m = R\sin\theta\,\dot\theta, \qquad (5)$$

$$v_{\rm m}^2 = R^2\,\dot\theta^2\left(\frac{M^2\cos^2\theta}{(M+m)^2} + \sin^2\theta\right). \qquad (6)$$

[1]In the absence of friction between the horizontal surface and the cylinder, the motion of the cylinder is the same whether the horizontal surface is at rest or in (horizontal) motion.

[2]A cylinder that rolls without slipping outside/inside a cylinder that rolls without slipping on a horizontal plane is discussed in http://kirkmcd.princeton.edu/examples/2cylinders.pdf
http://kirkmcd.princeton.edu/examples/2cylinders_in.pdf
These examples feature "ignorable" coordinates associated with nonintuitive conserved canonical momenta.

The total kinetic energy is,

$$T = T_{\rm cyl} + T_m = \frac{R^2\,\dot\theta^2}{2}\left[M\left(\frac{M^2\cos^2\theta}{(M+m)^2} + \sin^2\theta\right) + m\left(\frac{M^2\cos^2\theta}{(M+m)^2} + \sin^2\theta\right)\right]$$
$$= \frac{M^2R^2\,\dot\theta^2}{2(M+m)}\left(\cos^2\theta + \frac{(M+m)^2}{M^2}\sin^2\theta\right). \qquad (7)$$

and the (gravitational) potential energy, relative to height $R$ above the plane, is,

$$V = -mgR\cos\theta. \qquad (8)$$

Total energy $E = T + V$ is conserved in this problem, so for an initial configuration at rest with mass $m$ at angle $\theta_0$ we find that,

$$\frac{\dot\theta^2}{2} = \frac{m(M+m)g(\cos\theta - \cos\theta_0)}{M^2R}\left(\cos^2\theta + \frac{(M+m)^2}{M^2}\sin^2\theta\right)^{-1}. \qquad (9)$$

For what it's worth,

$$\ddot\theta = \frac{d}{d\theta}\left\{\frac{m(M+m)g(\cos\theta - \cos\theta_0)}{M^2R}\left(\cos^2\theta + \frac{(M+m)^2}{M^2}\sin^2\theta\right)^{-1}\right\}. \qquad (10)$$

The motion is oscillatory, and for small $\theta_0$ has the form,

$$\theta = \theta_0\cos\omega t, \qquad \dot\theta = -\omega\,\theta_0\sin\omega t, \qquad \left\langle\dot\theta^2\right\rangle = \frac{\omega^2\,\theta_0^2}{2}, \qquad (11)$$

while from eq. (9),

$$\left\langle\dot\theta^2\right\rangle \approx \frac{2m(M+m)g}{M^2R}\left\langle\cos\theta - \cos\theta_0\right\rangle \approx \frac{2m(M+m)g}{M^2R}\frac{\theta_0^2}{2}, \qquad (12)$$

and hence the angular frequency $\omega$ of the small oscillations is,

$$\omega \approx \sqrt{\frac{2m(M+m)g}{M^2R}}. \qquad (13)$$

If $m = 0$, there is no oscillation.

# Remarks

This problem was suggested by Vladimir Onoochin, based on a paper by Nivaldo Lemos, https://arxiv.org/abs/2111.06226.[3]

That paper supposes the cylinder rolls without slipping on a horizontal plane (at rest), and that mass $m$ slides without friction on the inside of the hollow cylinder of mass $M$. In this

[3] Also N.A. Lemos, *Breakdown of the connection between symmetries and conservation laws for semi-holonomic systems*, Am. J. Phys. **90**, 221 (2022), http://kirkmcd.princeton.edu/examples/mechanics/lemos_ajp_90_221_22.pdf.

case there is an "external" horizontal force on the system of masses $M$ and $m$, so its center of mass does not remain at rest and its horizontal momentum is not conserved. However, as Lemos' noted, the Lagrangian $\mathcal{L}$ for the cylinder + block of his example is independent of position $x$ of the center of the cylinder along the horizontal plane, so there exists a conserved canonical momentum $p_x = \partial\mathcal{L}/\partial\dot{x}$, which is not the total linear momentum of the system in this example.[4] He then called the independence of the Lagrangian on position $x$ "translation invariance" (as commonly done), and noted that examples of his type (which he called "semiholonomic") are exceptions to a naïve interpretation of Noether's theorem about conservation laws and "symmetries" to be that "translation invariance" implies conservation of linear momentum.[5] Lemos characterized this as a "breakdown", but it seems to the present author that Noether's theorem works well for his example.[6]

A possible moral is that the term "translation invariance" is subject to misinterpretation, as it is readily associated with other meanings than independence of a Lagrangian on a spatial coordinate. For example, if mass $m$ is subject to a force related by potential $V(x)$, the motion of the mass is of the same form if the mass and the source of the potential are both translated by any $\Delta x$, but this "translation invariance" does not mean that the linear momentum of the mass is conserved.

# A Appendix: Lemos' Example and the Routhian

In Lemos' example, https://arxiv.org/abs/2111.06226, the cylinder rolls without slipping on the horizontal plane, and mass $m$ slides without friction inside the (hollow) cylinder of mass $M$. The Lagrangian for this system is given by Lemos' eq. (5),

$$\mathcal{L} = T - V = \frac{(2M+m)\dot{x}^2}{2} + \frac{mR^2\,\dot{\theta}^2}{2} + mR\dot{x}\,\dot{\theta}\cos\theta + mgR\cos\theta. \tag{14}$$

The Lagrangian is independent of $x$ (which is called a cyclic or ignorable coordinate in the British literature), and the corresponding conserved canonical momentum is (Lemos' eq. (8)),

$$p_x = \frac{\partial\mathcal{L}}{\partial\dot{x}} = (2M+m)\dot{x} + mR\,\dot{\theta}\cos\theta = \text{const.} \tag{15}$$

If we suppose the system starts from rest, then $p_x = 0$ and,

$$\dot{x} = -\frac{mR\,\dot{\theta}\cos\theta}{2M+m}, \qquad x = A - \frac{mR\sin\theta}{2M+m}. \tag{16}$$

[4]There is no simple physical interpretation of this conserved momentum, but it is useful in quickly finding the equation of motion in $\theta$, as discussed in Appendix A below.

[5]Noether, https://arxiv.org/abs/physics/0503066, was concerned with the issue of conservation of energy in general relativity, and never mentioned momentum or "translation invariance". *For historical comments, see* https://arxiv.org/abs/physics/9807044. What is commonly called Noether's theorem in classical mechanics is that if a Lagrangian $\mathcal{L}$ is independent of coordinate $q$, then the canonical momentum $p_q = \partial\mathcal{L}/\partial\dot{q}$ is conserved, and if $\mathcal{L}$ is independent of time then the Hamiltonian $\mathcal{H} = \sum p_q\dot{q} - \mathcal{L})$ is conserved (but not necessarily the energy $T+V$), as was known to Hamilton.

See also https://en.wikipedia.org/wiki/Noethers_theorem.

[6]The first example in the present note is also "translation invariant" in the Lagrangian sense, but here the conserved canonical momentum is the total, horizontal, linear momentum of the system of masses $M$ and $m$, as discussed in Appendix B below.

If $x = 0$ is the initial coordinate of the center of the cylinder, and $\theta_0$ that of mass $m$,

$$A = \frac{mR\sin\theta_0}{2M + m} \approx \frac{mR\theta_0}{2M + m}\,, \tag{17}$$

where the approximation holds for small $\theta_0$.

In examples like this, with an ignorable coordinate, it may be advantageous to consider the Routhian $\mathcal{R}$,[7] which is a function of $(x, p_x, \theta, \dot\theta)$ rather than $(x, \dot x, \theta, \dot\theta)$,

$$\mathcal{R} = \dot x p_x - \mathcal{L}, \tag{18}$$

for which the equation of motion in $\theta$ is,

$$\frac{d}{dt}\frac{\partial\mathcal{R}}{\partial\dot\theta} = \frac{\partial\mathcal{R}}{\partial\theta}\,. \tag{19}$$

However, since $\partial\mathcal{R}/\partial\dot\theta = -\partial\mathcal{L}/\partial\dot\theta$ and $\partial\mathcal{R}/\partial\theta = -\partial\mathcal{L}/\partial\theta$, there is actually little advantage to use of the Routhian in this example.[8]

We now deduce the $\theta$ equation of motion directly from the Lagrangian (14),

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta} = mR^2\,\ddot\theta + mR\ddot x\cos\theta - mR\dot x\,\dot\theta\sin\theta = \frac{\partial\mathcal{L}}{\partial\theta} = -mR\dot x\,\dot\theta\sin\theta - mgR\sin\theta. \tag{22}$$

Note that if $m = 0$, there is no equation of motion in $\theta$ (and, of course, no motion in $\theta$).

From eq. (15) we have,

$$\dot p_x = (2M + m)\ddot x + mR\,\ddot\theta\cos\theta - mR\,\dot\theta^2\sin\theta = 0, \tag{23}$$

so the $\theta$ equation of motion can be written as,

$$mR^2\,\ddot\theta\left(1 - \frac{m\cos^2\theta}{2M + m}\right) + \frac{m^2R^2\,\dot\theta^2\sin 2\theta}{2(2M + m)} = -mgR\sin\theta. \tag{24}$$

For small oscillations of the form $\theta = \theta_0\cos\omega t$, eq. (24) implies that their angular frequency is,

$$\omega^2\left(\frac{2M}{2M + m} + \frac{m\,\theta_0^2\cos^2\omega t}{2M + m}\right) \approx \frac{g}{R}.\,, \tag{25}$$

$$\omega \approx \sqrt{\frac{(2M + m)g}{2MR}}\,, \tag{26}$$

[7] https://en.wikipedia.org/wiki/Routhian_mechanics

[8] The Routhian is perhaps prone to misuse, in that eq. (18), without replacing $\dot x$ by the form (16), would lead to,

$$\mathcal{R} = \frac{(2M + m)\dot x^2}{2} - \frac{mR^2\,\dot\theta^2}{2} - mgR\cos\theta, \tag{20}$$

and then eq. (19) would imply that,

$$mR^2\,\ddot\theta = -mgR\sin\theta, \tag{21}$$

which holds only if the cylinder is always at rest.

which is fairly close to that of a simple pendulum of length $R$. That is, the motion of the cylinder is slight, so the problem almost reduces to the case of mass $m$ sliding without friction inside a cylinder at rest.

However, as $m \to 0$, $\omega \to \sqrt{g/R}$, while for $m = 0$ there is no oscillation. This "discontinuous" behavior also holds for the case of mass $m$ sliding without friction inside a fixed cylinder.

## A.1 The Hamiltonian

Lemos' example involves friction, but the force of friction does no work, such that the energy $T + V$ is conserved.

For completeness, we note that the Lagrangian and Hamiltonian are independent of time, so the Hamiltonian a constant of the motion, which should be the energy.

For motion that starts from rest, $p_x = 0$,

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot\theta} = mR^2\dot\theta + mR\dot x \cos\theta = mR^2\dot\theta \left(1 - \frac{m\cos^2\theta}{2M + m}\right) \tag{27}$$

$$T = \frac{(2M+m)\dot x^2}{2} + \frac{mR^2\,\dot\theta^2}{2} + mR\dot x\,\dot\theta\cos\theta = \frac{mR^2\dot\theta}{2}\left(1 - \frac{m\cos^2\theta}{2M+m}\right) = \frac{p_\theta\dot\theta}{2}\,, \tag{28}$$

recalling eqs. (14) and (16). Hence,

$$\mathcal{H} = p_x\dot x + p_\theta\dot\theta - \mathcal{L} = p_\theta\dot\theta - T + V = \frac{p_\theta\dot\theta}{2} + V = T + V, \tag{29}$$

is the constant energy of the system of masses $M$ and $m$, as expected.

# B Appendix: Lagrangian Analysis of the First Example: No Friction, Mass $m$ Fixed to Mass $M$

In a Lagrangian analysis of the example on p. 1 above, we should not begin with eq. (1), which is a "Newtonian" insight. Rather, we use $x = x_{\text{cyl}}$ and $\theta$ as the two independent coordinates of the system of masses $M$ and $m$.

Then, the kinetic energy of the cylinder is simply,

$$T_{\text{cyl}} = \frac{M\dot x^2}{2} + \frac{MR^2\dot\theta^2}{2}\,. \tag{30}$$

The coordinates of mass $m$ are,

$$x_m = x + R\sin\theta, \qquad y_m = R(1 - \cos\theta), \tag{31}$$

$$\dot x_m = \dot x + R\cos\theta\,\dot\theta, \qquad \dot y_m = R\sin\theta\,\dot\theta, \tag{32}$$

$$v_{\text{m}}^2 = \dot x^2 + 2R\dot x\,\dot\theta\cos\theta + R^2\,\dot\theta^2. \tag{33}$$

The total kinetic energy is,

$$T = T_{\rm cyl} + T_m = \frac{(M+m)(\dot{x}^2 + R^2\dot{\theta}^2)}{2} + mR\dot{x}\,\dot{\theta}\cos\theta, \tag{34}$$

and the (gravitational) potential energy, relative to height $R$ above the plane, is,

$$V = -mgR\cos\theta. \tag{35}$$

The Lagrangian $\mathcal{L} = T - V$ does not depend on coordinate $x$, and hence there is a conserved canonical momentum,

$$p_x = \frac{\partial\mathcal{L}}{\partial\dot{x}} = \frac{\partial T}{\partial\dot{x}} = (M+m)\dot{x} + mR\,\dot{\theta}\cos\theta = M\dot{x} + m\dot{x}_m = \text{const.} \tag{36}$$

That is, the $x$-component of the ordinary linear momentum of the system is conserved, so we recover the Newtonian insight that led to eq. (1) above, and the rest of the previous analysis follows (more quickly than via continuation of the Lagrangian analysis).

# C Appendix: Other Variants

## C.1 No Friction, Mass $m$ Slides Freely

If there is no friction between the hollow cylinder and the horizontal plane, or between the cylinder and mass $m$, then the angular velocity of the cylinder never changes. Then, for a system that starts from rest with mass $m$ at angle $\theta_0$ to the vertical, the Lagrangian is simply,

$$\mathcal{L} = \frac{M\dot{x}^2}{2} + \frac{m\left(\dot{x}^2 + 2R\dot{x}\,\dot{\theta}\cos\theta + R^2\dot{\theta}^2\right)}{2} + mRg\cos\theta. \tag{37}$$

The horizontal coordinate $x$ of the center of the cylinder is "ignorable", so there is a conserved canonical momentum,

$$p_x = \frac{\partial\mathcal{L}}{\partial\dot{x}} = M\dot{x} + m\left(\dot{x} + R\,\dot{\theta}\cos\theta\right) = M\dot{x} + m\dot{x}_m, \tag{38}$$

which is just the $x$-component of the total linear momentum of the system (as expected since there is zero $x$-component of the external force on the system). For a system that starts from rest, $p_x = 0$, and,

$$\dot{x} = -\frac{mR\,\dot{\theta}\cos\theta}{M+m}, \qquad \ddot{x} = \frac{mR\left(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta\right)}{M+m}. \tag{39}$$

The $\theta$ equation of motion is,

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = mR^2\,\ddot{\theta} + mR\ddot{x}\cos\theta - mR\dot{x}\,\dot{\theta}\sin\theta = \frac{\partial\mathcal{L}}{\partial\theta} = -mR\dot{x}\,\dot{\theta}\sin\theta - mgR\sin\theta, \tag{40}$$

$$mR^2\,\ddot{\theta}\left(1 - \frac{m\cos^2\theta}{M+m}\right) + \frac{m^2R^2\,\dot{\theta}^2\sin 2\theta}{2(M+m)} = -mgR\sin\theta. \tag{41}$$

For small oscillations (with nonzero mass $m$) of the form $\theta = \theta_0 \cos\omega t$, eq. (41) implies that their angular frequency is,

$$\omega^2 \left( \frac{M}{M+m} + \frac{m\,\theta_0^2 \cos^2 \omega t}{M+m} \right) \approx \frac{g}{R} \,, \tag{42}$$

$$\omega \approx \sqrt{\frac{(M+m)g}{MR}} \,, \tag{43}$$

which is fairly close to that of a simple pendulum of length $R$. That is, the motion of the cylinder is slight, so the problem almost reduces to the case of mass $m$ sliding without friction inside a cylinder at rest.

As in Lemos' variant, $\omega \to \sqrt{g/R}$ as $m \to 0$, although there is no oscillation if $m = 0$.

## C.2 Mass $m$ Fixed to the Cylinder, which Rolls without Slipping

In the fourth variant, we suppose the cylinder rolls without slipping on the horizontal plane, while mass $m$ is fixed to a point on the cylinder. Then, the angular velocity $\dot\theta$ of the cylinder is related to its horizontal velocity $\dot x$ by the rolling constraint,

$$\dot x = -R\,\dot\theta, \qquad x = R(\theta_0 - \theta), \tag{44}$$

for a system that starts from rest with the center of the cylinder at $x = 0$ and mass $m$ at angle $\theta_0$.

The kinetic energy of the cylinder is,

$$T_{\rm cyl} = \frac{M\dot x^2}{2} + \frac{MR^2\,\dot\theta^2}{2} = M\dot x^2 = MR^2\,\dot\theta^2, \tag{45}$$

and the kinetic energy of mass $m$ is given by,

$$x_m = x + R\sin\theta, \qquad y_m = R(1 - \cos\theta), \tag{46}$$

$$\dot x_m = \dot x + R\,\dot\theta\cos\theta = R\,\dot\theta(\cos\theta - 1), \qquad \dot y_m = R\,\dot\theta\sin\theta, \tag{47}$$

$$T_m = mR^2\,\dot\theta^2(1 - \cos\theta). \tag{48}$$

The Lagrangian of the system is,

$$\mathcal{L} = R^2\,\dot\theta^2[M + m(1 - \cos\theta)] + mRg\cos\theta. \tag{49}$$

The $\theta$ equation of motion is,

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta} = 2R^2\,\ddot\theta[M + m(1 - \cos\theta)] + 2mR^2\,\dot\theta^2\sin\theta = \frac{\partial\mathcal{L}}{\partial\theta} = mR^2\,\dot\theta^2\sin\theta - mgR\sin\theta, \tag{50}$$

$$\ddot\theta\left(\frac{M}{m} + 1 - \cos\theta\right) + \dot\theta^2\sin\theta = -\frac{g}{2R}\sin\theta. \tag{51}$$

For small oscillations of the form $\theta = \theta_0\cos\omega t$, eq. (51) implies that their angular frequency is,

$$\omega^2 \approx \frac{mg}{2MR} \,, \qquad \omega \approx \sqrt{\frac{mg}{2MR}} \,. \tag{52}$$

If $m = 0$, there are, of course, no oscillations.