The Rod-Ring Paradox

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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1 Problem

As recently discussed in [1], a rod in relative motion to a ring along their line of centers makes "first contact" with the ring at different points along the rod in different frames of reference, if the rod is tilted with respect to the relative-velocity vector, and the rod and ring both lie in a common plane. Yet, it seems intuitive that the "first contact" is a single spacetime event.

What's going on here?

2 Solution

2.1 Rod and Ring Pass Through One Another

If we suppose that the rod and ring can pass through one another, then after the "first contact" at a single point, the rod and ring overlap at two points for a while, then only at one point at the moment of "last contact", after which the rod and ring no longer overlap.

Taking the relative velocity of the rod and ring to be in the x-direction, and the x-y plane to be their common plane, the two points of overlap (which have different x-coordinates) at some time t in one inertial frame are not simultaneous points of overlap in another frame in which the relative (x) velocity of the rod and ring is different.[1] The spacetime event of "first contact" in one inertial frame is member of a pair of overlap events in a different inertial frame, and hence is not an event of "first contact" in the latter frame.

That is, the "first contact" is not a unique spacetime event, but is frame dependent.

This was illustrated in Figs. 5 and 6 of [1] for a rod and ring with relative x-velocity $|\beta| = |v|/c = \sqrt{3}/2$, and the same extent in y. Also shown in blue in Fig. 5 is the position of the rod when it passes through the point of "first contact" in Fig. 6, and similarly in Fig. 6 for the position of the rod when it passes through the point of "first contact" in Fig. 5. In each case the blue line crosses the ring at two points.

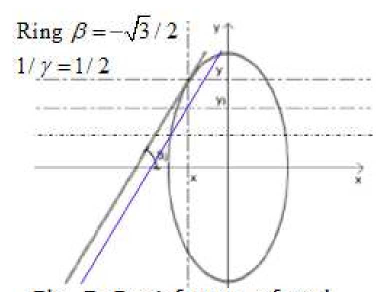


Fig. 5. Rest frame of rod

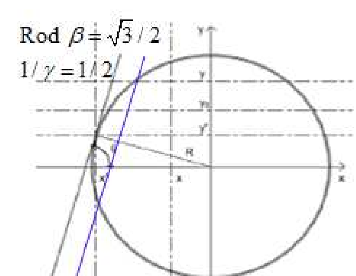


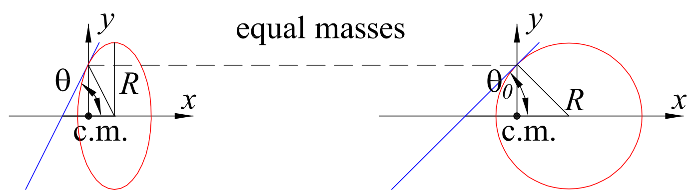
Fig. 6. Rest frame of ring

[1] This is an example of the relativity of simultaneity.

Unfortunately, it was ambiguously implied on p. 14 of [1][2] that the "first contact" is a single spacetime event, which is representative of the confusing discussion in that paper.[3]

2.2 Rod and Ring Collide and Stick Together

We also consider the case that the rod and ring are in the same plane, and stick together after they collide. Thereafter, the rod and ring deform and vibrate, dissipating their initial kinetic energy, until in the center-of-mass (c.m.) frame they end at rest and with their initial, rest-frame shapes. The figure below sketches this for equal masses of the rod and ring, where each had $|\beta| = |v|/c = \sqrt{3}/2$ and $1/\gamma = \sqrt{1-\beta^2} = 1/2$ in the c.m. frame before the collision, and the rod made angle $\theta_0 = 45°$ to the x-axis in its initial rest frame.



c.m. frame before collision,
rod β = √3/2 = - ring β, 1/γ = 1/2,
rod and ring have equal masses

c.m. frame after collision,
rod and ring are at rest
with initial, rest-frame shapes

There was no initial angular momentum in the c.m. frame, so the line of centers of the rod and ring is in the same direction before and after the collision. Then, the final configuration in the c.m. frame has the rod at the same angle θ_0 to the x-axis as in the initial rest frame of the rod.[4]

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[2] "However in the present case a similar paradox appears with only one event involved: the first point of contact of the shadows of the rod and the ring."

[3] Elsewhere in [1], there is awareness that "first contact" is a different "physical situation" in different frames.

[4] This simple result holds only for equal masses of the rod and ring. For unequal masses, $\gamma_{\rm rod}$ and $\gamma_{\rm ring}$ in the c.m. frame are different. The angle θ of the rod to the x-axis in the c.m. frame before the collision is related to the angle θ_0 of the rod in its rest frame by eq. (8) of [1], $\tan\theta = \gamma_{\rm rod}\tan\theta_0$, and the y-coordinate of the point of "first contact" with the ring in the c.m. frame follows from eq. (7) of [1] as, where R is the radius of the ring in its rest frame,

$$y = \frac{R\cos\theta}{\sqrt{1-\beta_{\rm ring}^2\sin^2\theta}} = \frac{R}{\sqrt{1+\gamma_{\rm rod}^2\tan^2\theta_0}\sqrt{1-\frac{\beta_{\rm ring}^2\gamma_{\rm rod}^2\tan^2\theta_0}{1+\gamma_{\rm rod}^2\tan^2\theta_0}}} = \frac{R}{\sqrt{1+\frac{\gamma_{\rm rod}^2}{\gamma_{\rm ring}^2}\tan^2\theta_0}}. \quad (1)$$

Only for $\gamma_{\rm rod} = \gamma_{\rm ring}$ in the c.m. frame (*i.e.*, only for equal masses) do we obtain the result $y = R\cos\theta_0$, which holds for the point of contact between the rod and ring when they are both at rest (as in the right figure above, and in Fig. 4 of [1]).

An example of the general case is sketched on the next page, with $\gamma_{\rm rod} = 4$ and $\gamma_{\rm ring} = 2$ in the c.m. frame. This implies that $m_{\rm ring} = m_{\rm rod}\sqrt{(\gamma_{\rm rod}^2-1)/(\gamma_{\rm ring}^2-1)} = \sqrt{3}\,m_{\rm rod}$. Then, the "first contact" is at $y = R/\sqrt{5} = 0.45R$ for $\theta_0 = 45°$. The c.m. is at distance $\sqrt{3}/(2+\sqrt{3}) = 0.53$ times the distance between the centers of the rod and ring, to the left of the center of the ring. With the c.m. at the origin in the c.m. frame, the x-coordinate of the point of contact is not $x = 0$, in general.

After the vibration of the rod and ring have ceased after the collision, and they have taken on their rest-frame shapes, the point of contact is no longer at the y-value of "first contact", as the line of centers of the rod and ring must remain along the x-axis.

References

[1] Ø. Grøn and E. Berntsen, *Rod-ring collision paradox* (March 17, 2022), https://arxiv.org/abs/2202.01057