Electric Potential of a Resistive Bead

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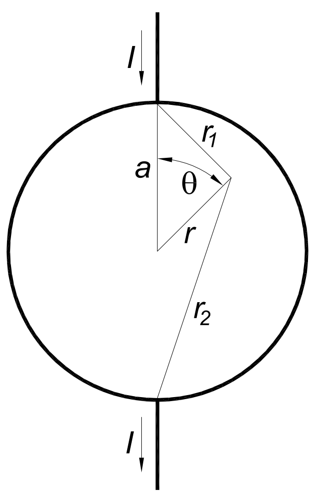
1 Problem

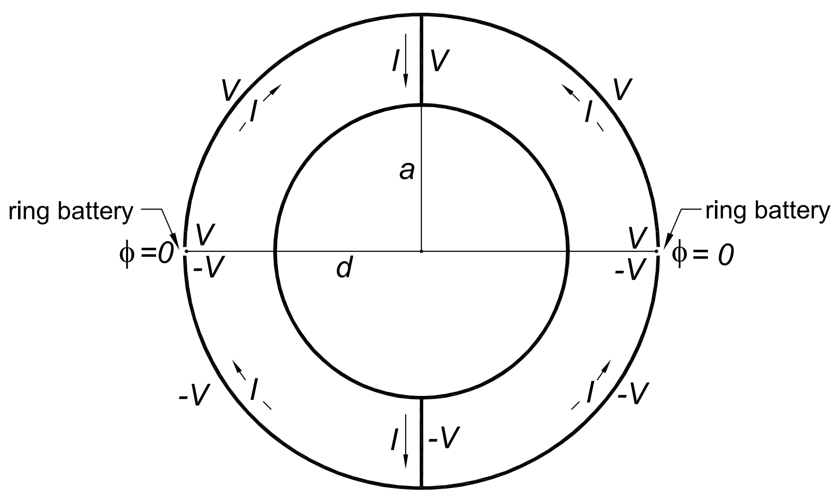
Obtain a Legendre series expansion for the potential inside a resistive bead of radius a and conductivity σ when a current I enters at one pole through a fine wire, and leaves through the other pole via a similar fine wire. Define the potential as $\phi = 0$ on the equator.

Refer to the expansion of $1/r$ given, for example, in eq. (3.38) of [2], to show that,

$$\begin{aligned} \phi(\mathbf{r}) &= \phi(r,\theta,\varphi) = \frac{I}{2\pi\sigma}\left[\frac{1}{r_1(\mathbf{r})} - \frac{1}{r_2(\mathbf{r})} + \frac{1}{2}\int_0^r \left(\frac{1}{r_1(\mathbf{r}')} - \frac{1}{r_2(\mathbf{r}')}\right)\frac{dr'}{r'}\right] \\ &= \frac{I}{2\pi\sigma}\left[\frac{1}{r_1(\mathbf{r})} - \frac{1}{r_1(\mathbf{r})} + \frac{1}{2a}\ln\frac{a + r\cos\theta + r_2(\mathbf{r})}{a - r\cos\theta + r_1(\mathbf{r})}\right], \qquad (1) \end{aligned}$$

using Dwight 380.111, where $r_{1,2}(\mathbf{r}) = \sqrt{r^2 \mp 2ar\cos\theta + a^2}$ is the distance from the "north" ("south") pole to the point $\mathbf{r} = (r,\theta,\varphi)$ in spherical coordinates.





Suppose the wires have radius $b \ll a$, and their surfaces of contact with the bead are equipotentials, to show that the resistance of the bead is that of a piece of wire roughly b long, if that wire also had conductivity σ

Hint: Express the radial current density at $r = a$ in terms of delta functions, $\delta(\cos\theta - 1)$ and $\delta(\cos\theta + 1)$.

Also deduce the potential outside the bead, and the surface charge density thereon, supposing that the fine wires are perfect conductors connected to perfect conducting hemispheres of radii $d > a$, which in turn are connected to a ring battery of voltage drop $2V$, where V is the potential at the "north" pole of the bead.

This problem appears in sec. 21 of [1], where eq. (1) is obtained by a different method than that suggested here.

2 Solution

2.1 Potential inside the Resistive Bead

Although current is flowing inside the resistive bead, its interior remains electrically neutral to a very good approximation.[1] Hence, the electromagnetic scalar potential ϕ satisfies Laplace's equation, $\nabla^2\phi = 0$.

We analyze the problem in spherical coordinates (r, θ, φ), with the origin at the center of the bead of radius a, and $\theta = 0$ and π at the points of contact with the wires. The problem has axial symmetry, so ϕ will be independent of φ. We require the potential to be well behaved at the origin, so it can be expressed in a Legendre series,

$$\phi(r < a) = \sum_{n=0}^{\infty} A_n \left(\frac{r}{a}\right)^n P_n(\cos\theta). \tag{2}$$

The convention that $\phi = 0$ at the equator, $\theta = \pi/2$, implies that $A_n = 0$ for n even. Therefore, we can write,

$$\phi(r < a) = \sum_{n \text{ odd}} A_n \left(\frac{r}{a}\right)^n P_n(\cos\theta). \tag{3}$$

To complete the solution inside the bead, we need a boundary condition on the potential ϕ at the surface of the sphere $r = a$. We know that the radial component of the current density, J_r is zero at the surface, except for the contact points where the current enters and exits. Since $\mathbf{J} = \sigma\mathbf{E} = -\sigma\boldsymbol{\nabla}\phi$, we obtain a condition on the derivative of the potential at the boundary,

$$\frac{\partial}{\partial r}\phi(r = a^-) = -E_r(r = a^-) = -\frac{J_r(r = a^-)}{\sigma}. \tag{4}$$

In the limit of very fine wires, the current density $J_r(r = a^-)$ is zero except at the poles, so we can express it in terms of Dirac δ functions. The current dI that crosses an annular region on the surface of the bead of angular extent $d\cos\theta$ centered on angle θ is given by,

$$dI = 2\pi a^2 J_r(a^-, \theta) d\cos\theta. \tag{5}$$

Current I enters at $\cos\theta = 1$, and exits at $\cos\theta = -1$. Hence, the form,

$$J_r(a^-, \theta) = \frac{I}{2\pi a^2}\left[-\delta(\cos\theta - 1) + \delta(\cos\theta + 1)\right]. \tag{6}$$

describes the entrance and exit currents upon integration of eq. (5).

Combining eqs. (3)-(4) and (6), we have,

$$\sum_{n \text{ odd}} \frac{nA_n}{a} P_n(\cos\theta) = \frac{I}{2\pi a^2\sigma}\left[\delta(\cos\theta - 1) - \delta(\cos\theta + 1)\right]. \tag{7}$$

[1]For a discussion of the slight departure from electrical neutrality of current-carrying conductors, see [3].

As usual, to evaluate the Fourier coefficients A_n, we multiply eq. (7) by $P_n(\cos\theta)$ and integrate over $d\cos\theta$ to find, recalling that $\int_{-1}^{1} P_m(x)P_n(x)\,dx = 2\,\delta_{mn}/(2n+1)$,

$$\frac{nA_n}{a}\int_{-1}^{1} P_n^2(\cos\theta)\,d\cos\theta = \frac{2nA_n}{(2n+1)a} = \frac{2I}{2\pi a^2\sigma}\,. \tag{8}$$

Thus, the Legendre series expansion for the potential is,

$$\phi(r<a,\theta) = \frac{I}{2\pi a\sigma}\sum_{n\ \mathrm{odd}}\left(2+\frac{1}{n}\right)\left(\frac{r}{a}\right)^n P_n(\cos\theta). \tag{9}$$

2.2 Current Density inside the Bead

Inside the resistive bead, the current density $\mathbf{J}$ is given by,

$$\mathbf{J} = \sigma\mathbf{E} = -\sigma\frac{\partial\phi}{\partial r}\,\hat{\mathbf{r}} - \frac{\sigma}{r}\frac{\partial\phi}{\partial\theta}\,\hat{\boldsymbol{\theta}} = -\sigma\frac{\partial\phi}{\partial r}\,\hat{\mathbf{r}} + \frac{\sigma\sin\theta}{r}\frac{\partial\phi}{\partial\cos\theta}\,\hat{\boldsymbol{\theta}}. \tag{10}$$

In the midplane, ($\theta = \pi/2$), the current density has only a θ-component ($-z$-component), with

$$J_z(r,\theta=\pi/2) = -\frac{I}{2\pi ar}\sum_{n\ \mathrm{odd}}\frac{2n+1}{n}\left(\frac{r}{a}\right)^n\frac{dP_n(0)}{d\cos\theta} \tag{11}$$
$$= \frac{I}{2\pi ar}\sum_{n\ \mathrm{odd}}\frac{(n+1)(2n+1)}{n}\left(\frac{r}{a}\right)^n P_{n+1}(0) = \frac{I}{\pi ar}\sum_{n=0}^{\infty}\frac{(n+1)(4n+3)}{2n+1}\left(\frac{r}{a}\right)^{2n+1} P_{2n+2}(0),$$

using eq. (14).[2]

The total current across the midplane is,

$$I_z(\theta=\pi/2) = 2\pi\int_0^a J_z(r,\theta=\pi/2)\,r\,dr \tag{16}$$
$$= \frac{I}{2a}\int_0^a\sum_{n=0}^{\infty}\frac{(n+1)(4n+3)}{2n+1}\left(\frac{r}{a}\right)^{2n+2} P_{2n+2}(0)\,dr = I\sum_{n=0}^{\infty}\frac{4n+3}{2n+1}P_{2n+2}(0) = -I,$$

as expected, based on numerical evaluation of eq. (16) for terms up to $n = N$ as shown in the figure below.[3]

[2]Some useful relations among the Legendre polynomials are, from eqs. (3.29) and (3.31) of [2],

$$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n+1}(x), \tag{12}$$

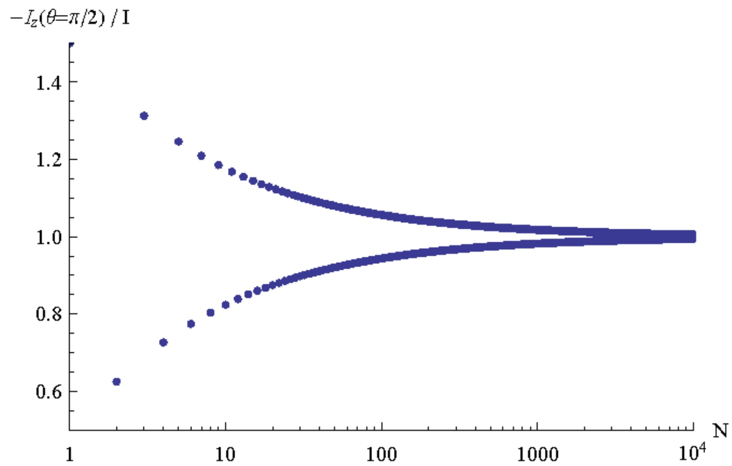
$$(1-x^2)\frac{dP_n(x)}{dx} = n[P_{n-1}(x) - xP_n(x)], \tag{13}$$

$$\frac{dP_n(x)}{dx} = nP_{n-1}(x) + x\frac{dP_{n-1}(x)}{dx} = -(n+1)P_{n+1}(x) + x\left((2n+1)P_n(x) + \frac{dP_{n-1}(x)}{dx}\right), \tag{14}$$

$$\int_{-1}^{1} xP_n(x)P_{m-1}(x)\,dx = \frac{2n}{(2n+1)(2n-1)}\,\delta_{mn}. \tag{15}$$

where $P'_n(x) = dP_n(x)/dx$.

[3]Due to Boris Ivetić, http://kirkmcd.princeton.edu/examples/resistive_bead_k3.nb.



The current density along the axis of the bead is,

$$J_z(r,\theta=0) = -\frac{I}{2\pi a^2}\sum_{n\ \mathrm{odd}}(2n+1)\left(\frac{r}{a}\right)^{n-1} = -\frac{I}{2\pi a^2}\left(3+7\frac{r^2}{a^2}+11\frac{r^4}{a^4}+\cdots\right), \tag{17}$$

which diverges at the poles ($r=a, \theta=0,\pi$), while that on the surface of the bead is,

$$\begin{aligned} J_\theta(a^-,\theta) &= -\frac{I\sin\theta}{2\pi a^2}\sum_{n\ \mathrm{odd}}\frac{2n+1}{n}P_n'(\cos\theta) \\ &= -\frac{I}{2\pi a^2\sin\theta}\sum_{n\ \mathrm{odd}}(2n+1)[P_{n-1}(\cos\theta)-\cos\theta P_n(\cos\theta)], \end{aligned} \tag{18}$$

using eq. (13), which also is ill behaved at the poles.[4]

These divergences result from the unphysical assumption that the wires have zero radius. For wires of finite radius b, the series (9) for the potential will be cut off at large n, as discussed further below, such that all fields and current densities are finite.

2.3 Closed Form for the Interior Potential

To express the series (9) in closed form, we utilize the expansion for the distance r_1 between the points $(a,0,\varphi)$ and $\mathbf{r}=(r,\theta,\varphi)$ given in eq. (3.38) of [2],

$$\frac{1}{r_1} = \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n P_n(\cos\theta), \tag{20}$$

Similarly, the distance r_2 between the points (a,π,φ) and (r,θ,φ) is,

$$\frac{1}{r_2} = \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n P_n(\cos(\theta-\pi)) = \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n P_n(-\cos\theta) = \frac{1}{a}\sum_{n=0}^{\infty}(-1)^n\left(\frac{r}{a}\right)^n P_n(\cos\theta). \tag{21}$$

[4]For $x=\cos\theta\approx 1$, $\sin\theta=\sqrt{1-x^2}\approx\sqrt{2(1-x)}$ and $P_n(x)=\sum_{m=0}^n [C_m^n]^2(x-1)^{n-m}(1+x)^m/2^n \approx 1-n^2(1-x)/2$, so that eq. (18) leads to,

$$J_\theta(a^-,x\approx 1) \approx -\frac{I}{2\pi a^2}\frac{\sqrt{1-x}}{2\sqrt{2}}\sum_{n\ \mathrm{odd}}(4n^2-1). \tag{19}$$

Hence,

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{2}{a} \sum_{n \text{ odd}} \left(\frac{r}{a}\right)^n P_n(\cos\theta). \tag{22}$$

It follows that, on integration along the radius from the origin to the point (r, θ),

$$\int_0^r \left(\frac{1}{r_1} - \frac{1}{r_2}\right) \frac{dr'}{r'} = \frac{2}{a} \sum_{n \text{ odd}} \frac{1}{n} \left(\frac{r}{a}\right)^n P_n(\cos\theta). \tag{23}$$

Then, eqs. (9) and (22)-(23) combine to give to the alternative form (1) for ϕ,

$$\begin{aligned} \phi(r < a, \theta, \varphi) &= \frac{I}{2\pi\sigma}\left[\frac{1}{r_1(\mathbf{r})} - \frac{1}{r_2(\mathbf{r})} + \frac{1}{2}\int_0^r \left(\frac{1}{r_1(\mathbf{r}')} - \frac{1}{r_2(\mathbf{r}')}\right) \frac{dr'}{r'}\right], \\ &= \frac{I}{2\pi\sigma}\left[\frac{1}{r_1(\mathbf{r})} - \frac{1}{r_1(\mathbf{r})} + \frac{1}{2a} \ln \frac{a + r\cos\theta + r_2(\mathbf{r})}{a - r\cos\theta + r_1(\mathbf{r})}\right], \qquad (1) \end{aligned}$$

where $\mathbf{r}' = (r', \theta, \varphi)$.

As we approach the "north" pole, $r_1 \to 0$, the first term in eq. (1) dominates. Similarly, near the "south" pole, $r_2 \to 0$, the second term dominates (we claim; details in eqs. (39)-(40) below). That is, the potential ϕ diverges at the poles for the case of very fine wires.

The electric field inside the bead has components,

$$\begin{aligned} E_r(\mathbf{r}) &= -\frac{\partial\phi}{\partial r} = \frac{I}{2\pi\sigma}\left[\frac{r - a\cos\theta}{r_1^3(\mathbf{r})} - \frac{r + a\cos\theta}{r_2^3(\mathbf{r})} + \frac{1}{2r\, r_1(\mathbf{r})} - \frac{1}{2r\, r_2(\mathbf{r})}\right], \qquad (24) \\ E_\theta(\mathbf{r}) &= \frac{\sin\theta}{r} \frac{\partial\phi}{\partial\cos\theta} \qquad (25) \\ &= \frac{I\sin\theta}{2\pi\sigma}\left[\frac{a}{r_1^3(\mathbf{r})} + \frac{a}{r_2^3(\mathbf{r})} + \frac{a + r_1(\mathbf{r})}{2ar_1[a - r\cos\theta + r_1(\mathbf{r})]} + \frac{a + r_2(\mathbf{r})}{2ar_2[a + r\cos\theta + r_2(\mathbf{r})]}\right]. \end{aligned}$$

The radial electric field (24) (and the radial current density $\mathbf{J} = \sigma\mathbf{E}$) vanishes on the surface of the bead, $r = a$, except at the poles, $\theta = 0, \pi$. The electric field on the midplane $\theta = \pi/2$ is,

$$E_\theta(\theta = \pi/2) = -E_z(\theta = \pi/2) = \frac{I}{2\pi\sigma}\left[\frac{2a}{(r^2 + a^2)^{3/2}} + \frac{1}{a(r^2 + a^2)^{1/2}}\right]. \tag{26}$$

The total current crossing the midplane is, using Dwight 201.01 and 201.03,

$$\begin{aligned} 2\pi \int_0^a |J_z(\theta = \pi/2)|\, r\, dr &= 2\pi\sigma \int_0^a |E_z(\theta = \pi/2)|\, r\, dr \\ = I \int_0^a r\, dr \left[\frac{2a}{(r^2 + a^2)^{3/2}} + \frac{1}{a(r^2 + a^2)^{1/2}}\right] &= I, \qquad (27) \end{aligned}$$

as expected.

2.4 Formal Solution for Wires of Radius b

This section was suggested by Boris Ivetić.

We can obtain a solution for wires of radius $b \ll a$, which make contact with the bead over a spherical cap of angle $\alpha = \sin^{-1} b/a \approx b/a$, if we suppose that the radial electric field on this cap is given by,

$$E_r(a^-, |\cos\theta| > \cos\alpha \approx 1 - b^2/2a^2) = \frac{J_r}{\sigma} \approx \mp \frac{I}{\pi b^2 \sigma}, \tag{28}$$

which is a good approximation for small b/a. As before, the radial electric field at the surface of the bead is zero outside the region of contact with the wires,

$$E_r(a^-, |\cos\theta| < \cos\alpha) = 0. \tag{29}$$

Now, eqs. (3)-(4), together with the boundary conditions (28)-(29), lead to the relation,

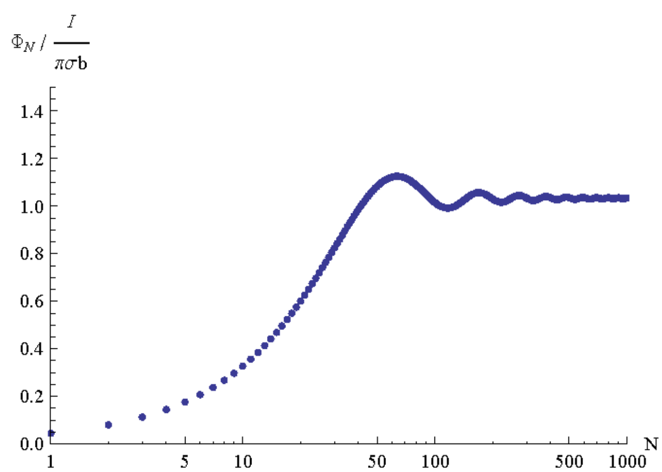
$$\sum_{n \text{ odd}} \frac{nA_n}{a} P_n(\cos\theta) = \pm \frac{I}{\pi b^2 \sigma} \begin{cases} 1 & (|\cos\theta| > \cos\alpha), \\ 0 & (|\cos\theta| < \cos\alpha). \end{cases} \tag{30}$$

Multiplying by P_n and integrating with respect to $x = \cos\theta$ yields, using eq. (12),

$$\frac{2nA_n}{(2n+1)a} = \frac{2I}{\pi b^2 \sigma} \int_{\cos\alpha}^1 P_n(x)\, dx = \frac{2I}{(2n+1)\pi b^2 \sigma} \int_{\cos\alpha}^1 (P'_{n+1}(x) - P'_{n-1}(x))\, dx, \tag{31}$$

$$A_n = \frac{Ia}{n\pi b^2 \sigma}(P_{n-1}(\cos\alpha) - P_{n+1}(\cos\alpha)). \tag{32}$$

A Mathematica evaluation of the potential, $\phi_N(r = a^-, \sin\theta = b/a = 0.03) = \sum_{n \text{ odd}}^{2N-1} A_n \left(\frac{r}{a}\right)^n P_n(\cos\theta) = \sum_{n=0}^{N} A_{2n+1} \left(\frac{r}{a}\right)^{2n+1} P_{2n+1}(\cos\theta)$, at the point of contact of the outer radius of the lead wire with the resistive bead, for $b/a = 0.03$, is shown below as a function of N, the number of terms computed.[5]



The numerical evaluation converges to $\approx 1.05 I/\pi\sigma b$ after roughly $N = 2a/b = 67$ terms, while sec. 2.2 indicates that the value should be $I/2\pi\sigma b$. *In my view, while the numerical*

[5]Due to Boris Ivetić, http://kirkmcd.princeton.edu/examples/resistive_bead_k1.nb.

evaluation converges, the asymptotic value so found is not necessarily accurate, as each $P_n(\cos\theta)$ is itself the result of a series expansion, also requiring $\gtrsim a/b$ terms for convergence, such that the overall numerical computation involves thousands of terms for $N > a/b$.[6,7]

So, we return to discussion of the forms (1) and (9) in sec. 2.5.

2.4.1 Current across the Midplane

For the case of wires with finite radius, $b/a = 0.03$, we use the potential with Fourier coefficients (32) to obtain the current density at the midplane, $\theta = \pi/2$, recalling eqs. (10) and (14),

$$\begin{aligned} J_z(r, \theta = \pi/2) &= -\frac{Ia}{\pi r b^2} \sum_{n \text{ odd}} \frac{1}{n} P_n'(0)(P_{n-1}(\cos\alpha) - P_{n+1}(\cos\alpha)) \left(\frac{r}{a}\right)^n \\ &= \frac{Ia}{\pi r b^2} \sum_{n \text{ odd}} \frac{n+1}{n} P_{n+1}(0)(P_{n-1}(\cos\alpha) - P_{n+1}(\cos\alpha)) \left(\frac{r}{a}\right)^n \\ &= \frac{Ia}{\pi r b^2} \sum_{n=0}^{\infty} \frac{2n+2}{2n+1} P_{2n+2}(0)(P_{2n}(\cos\alpha) - P_{2n+2}(\cos\alpha)) \left(\frac{r}{a}\right)^{2n+1}. \qquad (34) \end{aligned}$$

The corresponding total current flowing across the midplane is, using eq. (13),

$$\begin{aligned} I_z(\theta = \pi/2) &= 2\pi \int_0^a J_z\, r\, dr \\ &= 2I \frac{a^2}{b^2} \sum_{n=0}^{\infty} \frac{1}{2n+1} P_{2n}(0)(P_{2n}(\cos\alpha) - P_{2n+2}(\cos\alpha)). \qquad (35) \end{aligned}$$

A Mathematica evaluation of the current across the midplane, $I_N(\theta = \pi/2)$, is shown below for $b/a = 0.03$ as a function of N, the number of terms computed.[8]

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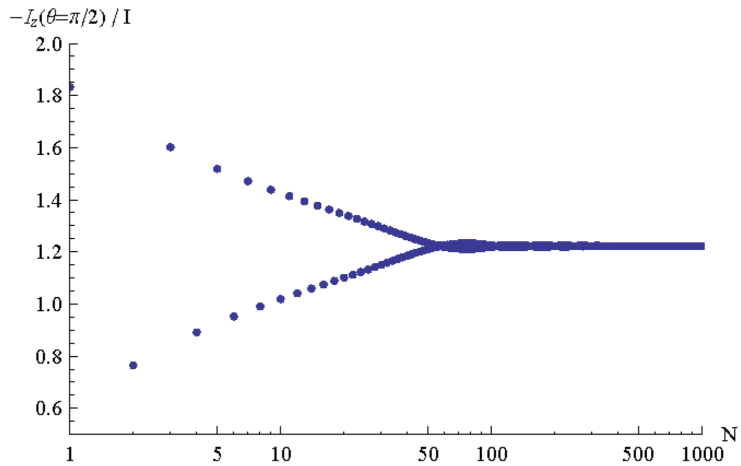
[6]For large enough n, $P_n(x)$ is oscillatory on the interval $[\cos\alpha, 1]$, so the integral $\int_{\cos\alpha}^1 P_n(x)\, dx$ goes to zero and the potential remains finite near the poles.

[7]In case of finite wires, $b/a = 0.03$, we can use the potential found above to obtain the current density along the axis of the bead,

$$J_z(r, \theta = 0) = -\sigma \frac{\partial\phi(r, \theta = 0)}{\partial r} = -\frac{I}{\pi b^2} \sum_{n \text{ odd}} (P_{n-1}(\cos\alpha) - P_{n+1}(\cos\alpha)) \left(\frac{r}{a}\right)^{n-1}. \qquad (33)$$

At the poles, $r = a$, $\theta = 0,\ \pi$, all terms in the series (33) cancel except for the very first, with $P_0(\cos\alpha) = 1$, which implies that $J_z(a, \theta = 0, \pi) = -I/\pi b^2$, as expected for lead wires of radius b.

[8]Due to Boris Ivetić, http://kirkmcd.princeton.edu/examples/resistive_bead_k4.nb.



The numerical evaluation converges to a value about 1.25 times that expected.

2.5 Ohm's Law for Wires of Radius b

When considering actual wires of radius $b \ll a$, we suppose that our solution (1) based on wires of zero radius, holds away from the region of contact between the wire and the bead. Indeed, we expect that the potential close to the wire, and outside the resistive bead, to be constant in planes perpendicular to the axis of the wire, so that the interface between the wire and the bead is an equipotential. This cuts off the formal divergence in eq. (1) near the poles.

In this way, the potential at the upper interface is obtained from (1) on putting $r_1 = b$ and neglecting all but the first term,

$$\phi_{\rm interface} \equiv V \approx \frac{I}{2\pi\sigma b}\,. \tag{36}$$

The potential difference across the bead is twice this,

$$\Delta V = 2V \approx \frac{I}{\pi\sigma b} = I\frac{b}{\sigma\pi b^2} \equiv IR. \tag{37}$$

Thus, the effective resistance of the bead is,

$$R \approx \frac{1}{\pi\sigma b} = \frac{b}{\sigma\pi b^2}, \tag{38}$$

which is also the resistance of a piece of wire of radius b, length b, and conductivity σ.

To verify the claim that the first term of eq. (1) dominates for small r_1, we consider the point $(r,\theta) = (a-b, 0)$ for $b \ll a$. Then, the first term of eq. (1) is $1/b$, and the second term is $1/(2a-b)$ which is negligible compared to the first. Inside the integral term of eq. (1), we have $R_1 = a - r$ and $R_2 = a + r$, so the integral is,

$$\int_0^{a-b} \left(\frac{1}{r_1} - \frac{1}{r_2}\right) d\ln r = \int_0^{a-b} \frac{2}{a^2 - r^2}\, dr = \frac{1}{a}\ln\frac{2a-b}{b} \approx \frac{1}{a}\ln\frac{2a}{b}\,. \tag{39}$$

The ratio of the integral term to the first term of (1) is therefore,

$$\frac{b}{2a}\ln\frac{2a}{b}\,, \tag{40}$$

which goes to zero as b becomes small.

2.6 Magnetic Field and Poynting Vector

The power dissipated by the resistive bead is, according to eq. (38),

$$P = I^2R \approx \frac{I^2}{\pi\sigma b}\,. \tag{41}$$

As a check on the solution (9) for the potential, we consider whether the dissipated power equals the integral of the Poynting flux, $\mathbf{S} = (c/4\pi)\mathbf{E}\times\mathbf{B}$ (in Gaussian units), normal to the surface of the bead, where c is the speed of light in vacuum.

For this, we need the magnetic field $\mathbf{B}$ at $r = a$, due to the electric currents in the problem. This field is azimuthal, because of the azimuthal symmetry of the problem. Then, the magnetic field at the surface of the resistive bead follows easily from Ampère's law,

$$2\pi r_\perp B_\varphi(r = a^-) = 2\pi a\sin\theta B_0 \ = -\frac{4\pi}{c}I, \qquad B_\varphi(r = a^-) = -\frac{2I}{c\,a\sin\theta}\,, \tag{42}$$

The radial component of the Poynting vector also depends on the electric field component,

$$E_\theta(r = a^- = -\frac{1}{a}\frac{\partial\phi(r = a^-)}{\partial\theta}. \tag{43}$$

Then, the radial component of the Poynting vector at the surface of the bead is,

$$\begin{aligned} S_r(r = a^-,\theta) &= \frac{c}{4\pi}E_\theta(r = a^-)B_\varphi(r = a^-) = \frac{I}{2\pi a^2}\frac{1}{\sin\theta}\frac{\partial\phi(r = a^-)}{\partial\theta} \\ &= -\frac{I}{2\pi a^2}\frac{\partial\phi(r = a^-)}{\partial\cos\theta} = -\frac{I}{2\pi a^2}\frac{I}{2\pi a\sigma}\sum_{n\ \mathrm{odd}}\left(2+\frac{1}{n}\right)P_n'(\cos\theta). \end{aligned} \tag{44}$$

The integral of the radial component of the Poynting vector over the surface of the bead is,

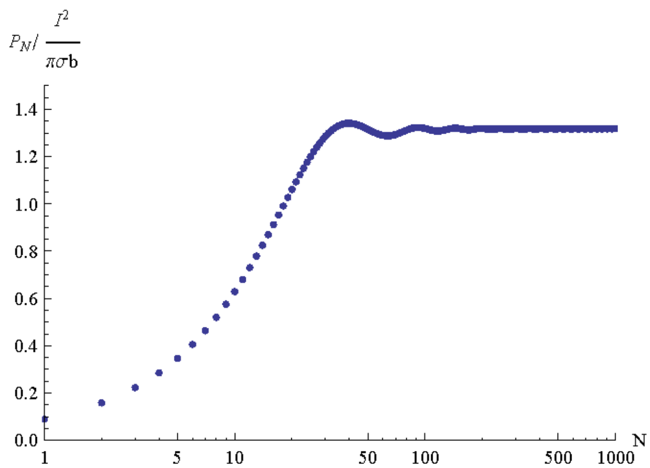
$$\begin{aligned} P_{\mathrm{into\ bead}} &= -2\pi a^2\int_{-1}^{1} S_r(r = a^-,\theta)\,d\cos\theta = \frac{I^2}{2\pi a\sigma}\sum_{n\ \mathrm{odd}}\left(2+\frac{1}{n}\right)\int_{-1}^{1} P_n'(\cos\theta)\,d\cos\theta \\ &= \frac{I^2}{\pi a\sigma}\sum_{n\ \mathrm{odd}}\left(2+\frac{1}{n}\right) \approx \frac{b}{a}I^2R\sum_{n\ \mathrm{odd}}\left(2+\frac{1}{n}\right). \end{aligned} \tag{45}$$

Formally, the result (45) diverges, which corresponds to infinite power dissipation at the points of contact of the wires with the bead, in the limit of zero radius of these wires. For wires of finite radius b, the power dissipated is finite, $P = I^2R$ for resistance R as approximated in eq. (38), but then the formal solutions (1) and (9) are only approximate. Since the sum of the first N terms of the series $\sum_{N\ \mathrm{odd}}(2+1/n)$ is roughly N, we infer that the form (9) for the potential inside the bead in case of wires of radius b is a reasonable approximation if we keep only the first $N \approx a/b$ terms.

2.6.1 Finite wire case

Following the same steps as in eqs (42)-(45) for the potential as calculated in sec. 2.4, the total power flowing into the bead is,

$$\begin{aligned} P_{\text{into bead}} &= -2\pi a^2 \int_{\cos(\alpha)}^{\cos(\pi-\alpha)} S_r(r=a^-,\theta)\, d\cos\theta \\ &= \frac{2I^2}{\pi b\sigma}\frac{a}{b}\sum_{n \text{ odd}} \frac{1}{n} P_n(\cos\alpha)\,(P_{n-1}(\cos\alpha) - P_{n+1}(\cos\alpha)) \\ &= \frac{2I^2}{\pi b\sigma}\frac{a}{b}\sum_{n=0}^{\infty} \frac{1}{2n+1} P_{2n+1}(\cos\alpha)\,(P_{2n}(\cos\alpha) - P_{2n+2}(\cos\alpha))\,. \qquad (46) \end{aligned}$$

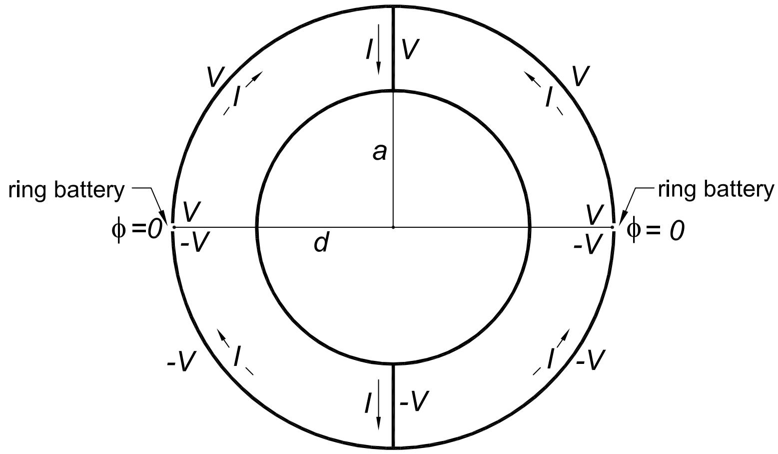


A Mathematica computation of eq. (46) for $b/a = 0.03$ is show above, as a function of the number N of terms kept in the summation.[9] The series appears to converge to $1.3I^2/\pi\sigma b$ rather than to $I^2R = I^2/2\pi\sigma b$ as expected from sec. 2.5. If the numerical result in sec. 2.4 were correct, we would expect eq. (46) to converge to $1.05I^2/\pi\sigma b$, which suggests that neither numerical result is highly accurate.

2.7 Potential outside the Resistive Bead

We now consider the problem outside the bead, for which one model is that the wires are perfect conductors extending from $r = a$ to distance d, where they are attached to perfectly conducting hemispheres of radius d, with a ring-shaped battery of potential difference $2V$ between them at location $(r,\theta) = (d, \pi/2)$.

[9]Due to Boris Ivetić, http://kirkmcd.princeton.edu/examples/resistive_bead_k2.nb.



In the region $a < r < d$ and outside the wires at $\theta = 0$ and π, the potential obeys $\nabla^2\phi = 0$, is azimuthally symmetric, and symmetric about the plane $\theta = \pi/2$, so it can be expanded as,

$$\phi(a < r < d) = \sum_{n \text{ odd}} \left[B_n \left(\frac{r}{a}\right)^n + C_n \left(\frac{a}{r}\right)^{n+1} \right] P_n(\cos\theta). \tag{47}$$

Continuity of the potential at $r = a$ requires, recalling eqs. (3) and (8), that,

$$B_n + C_n = A_n = \frac{(2n+1)I}{2\pi a n \sigma} = \frac{(2n+1)bV}{an}. \tag{48}$$

The constant potential V on the upper hemisphere requires that,

$$\phi(r = d, 0 < \cos\theta < 1) = V = \sum_{n \text{ odd}} \left[B_n \left(\frac{d}{a}\right)^n + C_n \left(\frac{a}{d}\right)^{n+1} \right] P_n(\cos\theta), \tag{49}$$

and the constant potential V on the upper wire requires that,

$$\phi(a < r < d, \theta = 0) = V = \sum_{n \text{ odd}} \left[B_n \left(\frac{r}{a}\right)^n + C_n \left(\frac{a}{r}\right)^{n+1} \right]. \tag{50}$$

If we multiply eq. (49) by $P_n(\cos\theta) = P_n(x)$ and integrate over x from 0 to 1, we obtain (using Wolfram Alpha with `integrate Legendre P(n,x) from x=0 to 1`),[10]

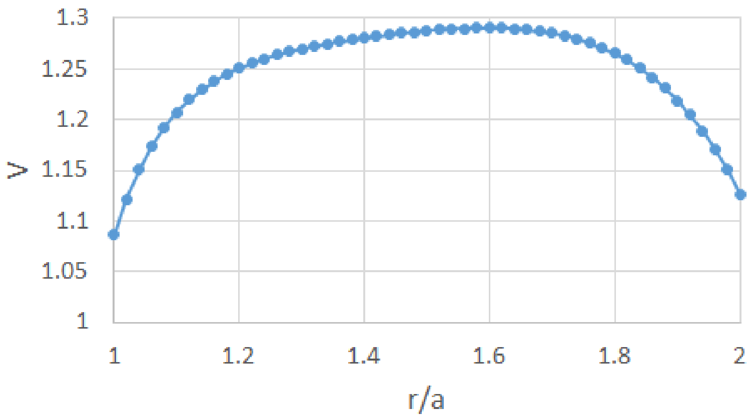
$$V \int_0^1 P_n(x)\, dx = \frac{\sqrt{\pi}\, V}{2\Gamma(1 - \frac{n}{2})\Gamma(\frac{n+3}{2})} \equiv K_n V = \frac{1}{2n+1} \left[B_n \left(\frac{d}{a}\right)^n + C_n \left(\frac{a}{d}\right)^{n+1} \right], \tag{51}$$

$K_1 = 1/2$, $K_3 = -1/8$, $K_5 = 1/16$, $K_7 = -5/128$, $K_9 = 7/256$, $K_{11} = -21/1024$, ... Combining this with eq. (48), we find,

$$B_n = \frac{(2n+1)V}{(\frac{d}{a})^n - (\frac{a}{d})^{n+1}} \left[K_n - \frac{b}{an} \left(\frac{a}{d}\right)^{n+1} \right], \qquad C_n = \frac{(2n+1)V}{(\frac{d}{a})^n - (\frac{a}{d})^{n+1}} \left[\frac{b}{an} \left(\frac{d}{a}\right)^n - K_n \right]. \tag{52}$$

[10] Jackson uses Rodrigues' formula and integration by parts n times to find $K_n = (-1/2)^{(n-1)/2}(n-2)!!/2[(n+1)/2]!$, his eq. (3.26) [2].

It is not obvious how well the B_n and C_n of eq. (52) satisfy the condition (50), but a numerical example suggests that they do so. For example, suppose that $d = 2a$, $b/a = 0.03$ and $V = 1$. Then, using only the first six terms of eq. (50) yields the following plot,[11]



The potential outside the perfectly conducting hemispheres, which are held at potentials $\pm V$*, is given in eq. (3.36) of [2]. In our notation,*

$$\phi(r > d) = V \sum_{n \text{ odd}} (2n+1) K_n \left(\frac{d}{r}\right)^{n+1} P_n(\cos\theta). \tag{53}$$

2.8 Surface Charge Density on the Resistive Bead

We also infer that the surface $r = a$ of the bead supports electric charge density,

$$\begin{aligned} \varsigma(\theta) &= \frac{E_r(r = a^+) - E_r(r = a^-)}{4\pi} = -\frac{\partial}{\partial r}[\phi(r = a^+) - \phi(r = a^-)] \\ &= \sum_{n \text{ odd}} \frac{-nB_n + (n+1)C_n + nA_n}{a} P_n(\cos\theta) = \sum_{n \text{ odd}} \frac{(2n+1)C_n}{a} P_n(\cos\theta). \end{aligned} \tag{54}$$

This illustrates the general result that current-carrying conductors (of finite conductivity σ) have nonzero surface charge density, as needed to shape the electric field $\mathbf{E} = \mathbf{J}/\sigma$ which drives the current inside the conductor [4].

For completeness, we compute the linear charge density λ *on the lead wires, and the surface charge density* ς *on the inside and outside surfaces of the hemispheres at* $r = d$*,*

$$\begin{aligned} \lambda(a < r < d, \theta = 0, \pi) &= 2\pi b\,\varsigma = \frac{b}{2} E_\theta = -\frac{b}{2r}\frac{\partial\phi}{\partial\theta} \\ &= \frac{b}{2r}\sin\theta \sum_{n \text{ odd}} \left[B_n \left(\frac{r}{a}\right)^n + C_n \left(\frac{a}{r}\right)^{n+1}\right] P_n'(\cos\theta) \\ &\approx \pm\frac{b^2}{2r^2} \sum_{n \text{ odd}} \left[B_n \left(\frac{r}{a}\right)^n + C_n \left(\frac{a}{r}\right)^{n+1}\right] P_n'(1) \end{aligned}$$

[11] In the limit that $d \gg a$, $B_n \to 0$, $C_n \to A_n$, and eq. (50) would imply that $V \to 0$. That is, to satisfy all three constraints (47), (49) and (50) on the potential $\phi(a < r < d)$, we need B_n nonzero, and d/a finite.

$$
\begin{aligned}
&\approx \pm\frac{b^2}{4r^2}\sum_{n\ \mathrm{odd}} n(n+1)\left[B_n\left(\frac{r}{a}\right)^n + C_n\left(\frac{a}{r}\right)^{n+1}\right], \qquad (55)\\
\varsigma(r=d^-) &= -\frac{E_r(r=d^-)}{4\pi} = \frac{1}{4\pi}\frac{\partial\phi(r=d^-)}{\partial r}\\
&= \frac{1}{4\pi d}\sum_{n\ \mathrm{odd}}\left[nB_n\left(\frac{d}{a}\right)^n - (n+1)C_n\left(\frac{a}{d}\right)^{n+1}\right]P_n(\cos\theta), \qquad (56)\\
\varsigma(r=d^+) &= -\frac{1}{4\pi}\frac{\partial\phi(r=d^+)}{\partial r} = \frac{V}{4\pi d}\sum_{n\ \mathrm{odd}}(n+1)(2n+1)K_nP_n(\cos\theta), \qquad (57)
\end{aligned}
$$

where $P'_n(1) = dP_n(x=1)/dx = n(n+1)/2$, and $\sin\theta \approx b/r$ for points on the surface of the lead wires.

A Appendix: Resistive Spherical Shell

This Appendix was suggested by Boris Ivetić.

We can also consider the case of a resistive spherical shell of outer radius a and thickness $\epsilon \ll a$.

A.1 Surface Current and Total Resistance of the Shell

The surface current density $\mathbf{K}$ (per unit length) on a resistive shell has only a θ-component, related by conservation of charge flowing across rings of circumference $2\pi a\sin\theta$ at angles θ as,

$$
K_\theta(\theta) = \frac{I}{2\pi a\sin\theta}\,. \qquad (58)
$$

To estimate the electrical resistance R of the spherical shell, we note that an annulus of extent $a\,d\theta$ about angle θ, with circumference $C = 2\pi a\sin\theta$, has resistance $dR = a\,d\theta/\sigma_S C = d\theta/2\pi\sigma_S\sin\theta$ to the surface current K_θ. To find the total resistance, we integrate dR from $\theta = b/a$ to $\pi - b/a$, supposing the current enters and exits the shell through wires of radius $b \ll a$. Then,

$$
R \approx 2\int_{b/a}^{\pi/2} dR = \frac{1}{\pi\sigma_S}\int_{b/a}^{\pi/2}\frac{d\theta}{\sin\theta} \approx -\frac{1}{\pi\sigma_S}\ln\frac{b}{2a}\,, \qquad (59)
$$

which is very large for small b/a, in contrast to the result (38) for a solid resistive bead (of volume conductivity σ).

A.2 Potential inside the Shell

The resistive shell supports charge densities on both of its surfaces, which in the limit of zero thickness ϵ we suppose is a single surface density $\varsigma(a,\theta)$. Then, the potential for $r < a-\epsilon \approx a$ (as well at that for $a < r < d$ and $r > d$) obeys $\nabla^2\phi = 0$, so we can again seek a potential in spherical coordinates that is independent of azimuth φ, and symmetric about the plane $\theta = 0$, with the form (3),

$$\phi(r < a) = \sum_{n \text{ odd}} A_n \left(\frac{r}{a}\right)^n P_n(\cos\theta). \tag{3}$$

This surface current is driven by the electric field E_θ inside the resistive shell according to Ohm's law, which we take to have the form,

$$K_\theta = \sigma_S E_\theta(r = a^-) = -\sigma \frac{\partial\phi(r = a^-)}{\partial\theta} = \sigma_S \sin\theta \frac{\partial\phi(r = a^-)}{\partial\cos\theta} = \sigma_S \sin\theta \sum_{n \text{ odd}} A_n P'_n(\cos\theta). \tag{60}$$

where σ_S is the surface conductivity.[12] Then, recalling eq. (58) and using eq. (13),

$$(1 - \cos^2\theta) \sum_{n \text{ odd}} A_n P'_n(\cos\theta) = \sum_{n \text{ odd}} n A_n [P_{n-1}(\cos\theta) - \cos\theta P_n(\cos\theta)] = \frac{I}{2\pi a \sigma_S}. \tag{61}$$

To find A_1, we simply integrate eq. (61) with respect to $\cos\theta$ from -1 to 1,

$$A_1 \int_{-1}^{1} (1 - \cos^2\theta)\, d\cos\theta = \frac{4A_1}{3} = \frac{I}{\pi a \sigma_S}, \qquad A_1 = \frac{3I}{4\pi a \sigma_S}. \tag{62}$$

For $n > 1$, we multiply eq. (61) by $P_{n-1}(\cos\theta)$ and integrate to find, using eq. (13),

$$\frac{2n(n+1)}{(2n-1)(2n+1)} A_n = \frac{2(n-1)(n-2)}{(2n-1)(2n-3)} A_{n-2}, \qquad A_n = \frac{(n-1)(n-2)(2n+1)}{n(n+1)(2n-3)} A_{n-2}. \tag{63}$$

Thus, $A_3 = 7A_1/18$, $A_5 = 11A_1/45$, $A_7 = 5A_1/28$, ..., with $A_n \approx (1 - 2/n)A_{n-2}$ for large n.

The potential is divergent at the poles of the shell, which divergence is suppressed in practice by the finite radius b of the lead wires, which implies that $A_n \to 0$ for n large enough that $P_n(\cos\theta)$ is oscillatory on the interval $0 < \theta < b/a$.

References

[1] L.D. Landau, E.M. Lifshitz and L.P. Pitaevskii, *Electrodynamics of Continuous Media*, 2nd ed. (Butterworth-Heinemann, 1984), http://kirkmcd.princeton.edu/examples/EM/landau_ecm2.pdf

[2] J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, 1999), http://kirkmcd.princeton.edu/examples/EM/jackson_ce3_99.pdf

[3] K.T. McDonald, *Charge Density in a Current-Carrying Wire* (Dec. 23, 2010), http://kirkmcd.princeton.edu/examples/wire.pdf

[4] J.D. Jackson, *Surface charges on circuit wires and resistors play three roles*, Am. J. Phys. **64**, 855 (1996), http://kirkmcd.princeton.edu/examples/EM/jackson_ajp_64_855_96.pdf

[12] Strictly, we should seek a potential inside the shell, for $a - \epsilon < r < a$, that leads to zero radial electric field E_r inside this shell. However, E_θ is continuous across the surface of the shell, so to the extent that we restrict our attention to E_θ in and around the shell, we can use the potential for $r < a - \epsilon$ in eq. (60).