The Relativity of Acceleration

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1 Problem

Acceleration is not often considered in special relativity,[1,2] but it can be,[3] as in the following problem.

A set of pointlike objects, each with initial velocity $\mathbf{v}_0 = (u, 0, 0)$, initially moving according to $\mathbf{x}_i(t) = (i\,L + u\,t, 0, 0)$ in one inertial frame, begin accelerating at a constant value $\mathbf{a} = (a, 0, 0)$ in the first frame at time $t'_{i,0} = 0$ in a second inertial frame that has velocity $\mathbf{v} = (v, 0, 0)$ with respect to the first.[4] Both a and v are positive.

What are the subsequent histories of these objects according to observers in these two frames? Can these objects collide with one another for any values of a, L, u and v?

2 Solution

See the Appendix below for a general discussion of the Lorentz transformation of acceleration.

A constant acceleration a cannot be sustained indefinitely, as the speed $u + a\Delta t$ of an accelerated object in the first frame would eventually exceed the speed of light c, which is impossible. So, this problem concerns only time intervals Δt of acceleration small enough that all speeds are less than c.

While the objects start accelerating simultaneously in the second frame, they do not start simultaneously in the first frame, due to the so-called relativity of simultaneity. This problem will illustrate that when the acceleration is constant in the first frame this is not so in the second frame, which effect could be called the relativity of acceleration.

Since all motion in this problem involves only the x-coordinates, it is sufficient to consider the Lorentz transformations of only the x-coordinate and time between the two frames,

$$x' = \gamma(x - v\,t), \qquad t' = \gamma(t - v\,x/c^2), \tag{1}$$

$$x = \gamma(x' + v\,t'), \qquad t = \gamma(t' + v\,x'/c^2), \tag{2}$$

$$\text{where} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}\,. \tag{3}$$

The theory of relativity is based, in part, on the premise that there is only a single

[1] One reason for this is that special relativity is often illustrated by "rods", that can be regarded as rigid bodies in inertial frames, but the acceleration of a rigid body is inconsistent with special relativity [1] (except for constant acceleration in the rest frame of the body). https://en.wikipedia.org/wiki/Born_rigidity

[2] Acceleration in an $\mathbf{F} = m\,\mathbf{a}$ context was mentioned in the final section of Einstein's first relativity paper [2], as reviewed in Appendix A below.

[3] Pedagogic discussions of acceleration in special relativity include [3]-[18].

[4] That is, the onset of the acceleration is simultaneous in the second inertial frame.

"reality", but different observers can describe this differently.[5] For example, the start of acceleration of object i is an "event", described by coordinates $(x_{i,0}, t_{i,0})$ in the first frame and coordinates $(x'_{i,0}, t'_{i,0})$ in the second frame (where $t'_{i,0} = 0$ for all i). These two sets of coordinates are related by the Lorentz transformations (1)-(2). However, there is only one object with index i, not different objects labeled by index i in different frames.

A collision between two objects, say i and $i-1$, would be an "event", described by coordinates $(x_{i:i-1}, t_{i:i-1})$ in the first frame and coordinates $(x'_{i:i-1}, t'_{i:i-1})$ in the second frame. Again, these two sets of coordinates would be related by the Lorentz transformations (1)-(2). However, there is only one collision between two objects, not different collisions in different frames. That is, if a collision occurs in one frame, it occurs in all, or if there is no collision in one frame there is no collision in any other.

In the present example, we will find no collisions in either frame, by somewhat different arguments in the two frames.

2.1 History during the Acceleration in the First Frame

Before time $t'_{i,0} = 0$ in the second frame, all objects i are in uniform motion, $x_i(t) = i\,L + u\,t$, in the first frame.

These objects start accelerating in the first frame at times $t_{i,0}$ that can be determined from the time transformation of eq. (1),

$$t_{i,0} = \frac{v\,x_{i,0}}{c^2} + \frac{t'_{i,0}}{\gamma} = \frac{v\,x_{i,0}}{c^2} = \frac{v(i\,L + u\,t_{i,0})}{c^2}\,, \qquad t_{i,0} = \frac{i\,v\,L}{c^2(1 - u\,v/c^2)}\,. \tag{4}$$

The acceleration motion of object i in the first frame is given by,

$$v_i(t) = u + a(t - t_{i,0}), \quad x_i(t) = x_{i,0} + u\,(t - t_{i,0}) + \frac{a}{2}\,(t - t_{i,0})^2 \qquad (t_{i,0} < t < t_{i,f})\,, \tag{5}$$

where $t_{i,f} = t_{i,0} + (c - u)/a$ such that $v_i \le c$ always.

That is, under the assumption of constant acceleration a in the first frame, object i (if it did not collide with any other object) would reach the speed of light (after infinite expenditure of energy by the accelerating mechanism) at the "final" time $t_{i,f}$ and "final" position $x_{i,f}$ related by,

$$t_{i,f} = t_{i,0} + \frac{c - u}{a}\,, \qquad x_{i,f} = x_{i,0} + u\,(t_{i,f} - t_{i,0}) + \frac{a}{2}\,(t_{i,f} - t_{i,0})^2\,. \tag{6}$$

2.1.1 Do the Objects Collide with One Another?

However, the motion (5) actually holds only if object i does not collide with either of its nearest neighbors $i-1$ and $i+1$ while accelerating.

Supposing that there is no limit to the duration of the acceleration, objects i and $i-1$ collide at time $t_{i:i-1}$ related by $x_i(t_{i:i-1}) = x_{i-1}(t_{i:i-1})$,

$$x_{i,0} + u\,(t_{i:i-1} - t_{i,0}) + \frac{a}{2}\,(t_{i:i-1} - t_{i,0})^2 = x_{i-1,0} + u\,(t_{i:i-1} - t_{i-1,0}) + \frac{a}{2}\,(t_{i:i-1} - t_{i-1,0})^2\,, \tag{7}$$

[5] This premise contrasts with that of the "multiverse" interpretation of quantum theory in which different observers experience different events in different universes.

$$x_{i,0} - x_{i-1,0} + u(t_{i-1,0} - t_{i,0}) + \frac{a}{2}\left[2t_{i:i-1}(t_{i-1,0} - t_{i,0}) + t_{i,0}^2 - t_{i-1,0}^2\right] = 0, \quad (8)$$

$$t_{i:i-1} = \frac{x_{i,0} - x_{i-1,0}}{a(t_{i,0} - t_{i-1,0})} - \frac{u}{a} + \frac{t_{i,0} + t_{i-1,0}}{2}, \quad (9)$$

$$t_{i:i-1} = t_{i,0} + \frac{c^2}{a\,v}\left(1 - \frac{u\,v}{c^2}\right) - \frac{t_{i,0} - t_{i-1,0}}{2} = t_{i-1,0} + \frac{c^2}{a\,v}\left(1 - \frac{u\,v}{c^2}\right) + \frac{t_{i,0} - t_{i-1,0}}{2}. \quad (10)$$

In particular, the supposed collision occurs at time,

$$\begin{aligned} t_{i:i-1} &= t_{i-1,0} + \frac{c-u}{a} + \frac{c^2}{a\,v}\left(1 - \frac{u\,v}{c^2}\right) - \frac{c-u}{a} + \frac{t_{i,0} - t_{i-1,0}}{2} \\ &= t_{i-1,f} + \frac{c}{a}\left(\frac{c}{v} - 1\right) + \frac{v\,L}{2c^2(1 - u\,v/c^2)} > t_{i-1,f}, \end{aligned} \quad (11)$$

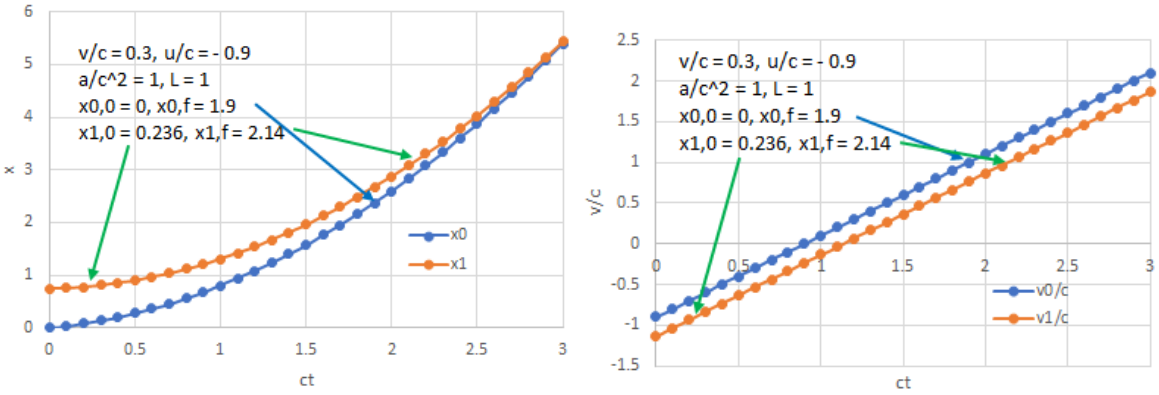
which is after time $t_{i-1,f}$ when object $i-1$ stopped accelerating (its velocity having reached the speed of light), independent of the initial velocity u. Hence, object $i-1$ would actually never catch up to object i, and no collision occurs.

Another way to see this is to consider the velocity of object $i-1$ at the time of the supposed collision,

$$v_{i-1}(t_{i:i-1}) = u + a(t_{i:i-1} - t_{i-1,0}) = u + \frac{c^2}{v}\left(1 - \frac{u\,v}{c^2}\right) + \frac{a\,v\,L}{2c^2(1 - \frac{u\,v}{c^2})} = \frac{c^2}{v} + \frac{a\,v\,L}{2c^2(1 - \frac{u\,v}{c^2})}, \quad (12)$$

which is greater than the speed c of light for any value of u.

The figures below shows $x_i(t)$ and $v_i(i)/c$ for $i = 0$ and 1, for parameters $L = 1$, $a/c^2 = 1$, $v/c = 0.3$ and $u/c = -0/9$.[6] While the objects formally collide at $c\,t = 3$, both of their velocities are greater than c at this time. That is, the collision did not actually occur.



2.2 History during the Acceleration in the Second Frame

Since no collisions between objects i occur in the first frame for any initial velocity u, we restrict the rest of this note to the case $u = 0$, which simplifies the algebra.[7]

[6] http://kirkmcd.princeton.edu/examples/rel_accel.xlsx

[7] We display one result in the second frame for arbitrary initial velocity u in the first frame of the objects, namely, the separation $L' = x'_{i,0} - x'_{i-1,0}$ between adjacent objects in the second frame at time $t'_i = 0$. Recall from eq. (4) that the coordinates of object i at time $t'_i = 0$ are $x_{i,0} = i\,L + u\,t_{i,0}$, $t_{i,0} = i\,v\,L/c^2(1 - u\,v/c^2)$. The

In the second frame, which has velocity v relative to the first, all objects i start accelerating at time $t'_{i,0} = 0$, and positions $x'_{i,0}$ that can be computed from the x-transformation of eq. (2),

$$x'_{i,0} = \frac{x_{i,0}}{\gamma} - v\, t'_{i,0} = \frac{x_{i,0}}{\gamma} = \frac{i\, L}{\gamma}\,, \tag{13}$$

which we recognize as an example of the Lorentz contraction of objects in motion with respect to an observer (here in the second frame).[8] The initial velocity of object i in the second frame is, of course, $v_{i,0} = -v$ (for $u = 0$).

The "final" time and position of object i (in the absence of any other objects) in the second frame, when and where its speed would reach c, can be computed using the Lorentz transformations (1), along with eq. (6),[9]

$$x'_{i,f} = \gamma(x_{i,f} - v\, t_{i,f}) = \gamma\left(i\, L + \frac{c^2}{2a} - \frac{i\, v^2 L}{c^2} - \frac{c\, v}{a}\right) = \frac{i\, L}{\gamma} + \frac{\gamma\, c}{a}\left(\frac{c}{2} - v\right), \tag{16}$$

$$t'_{i,f} = \gamma\left(t_{i,f} - \frac{v\, x_{i,f}}{c^2}\right) = \gamma\left(\frac{i\, v\, L}{c^2} + \frac{c}{a} - \frac{i\, v\, L}{c^2} - \frac{v}{2a}\right) = \frac{\gamma}{a}\left(c - \frac{v}{2}\right). \tag{17}$$

The "final" velocity of object i is, of course, $v'_{i,f} = c$.

2.2.1 Is the Acceleration of Object i Constant in the Second Frame?

If the motion object i in the second frame were that of constant acceleration a'_i, its velocity $v'_i(t')$ would have the time dependence during the interval $[t'_{i,0}, t'_{i,f}] = [0, t'_{i,f}]$,

$$v'_i(t') = v'_{i,0} + a'_i(t' - t'_{i,0}) = -v + a'_i t', \tag{18}$$

Lorentz transformation (1) yields $x'_{i,0} = \gamma(x_{i,0} - v t_{i,0}) = \gamma(i\, L + (u-v)t_{i,0}) = \gamma i\, l[1 + (u-v)v/c^2(1 - u\, v/c^2)] = \gamma i\, L(1 - v^2/c^2)/(1 - u\, v/c^2) = iL/\gamma(1 - u\, v/c^2)$. Hence, $L' = x'_{i,0} - x'_{i-1,0} = L/\gamma(1 - u\, v/c^2)$.

In a third frame, denoted by *, in which the objects are initially at rest, the initial separation between adjacent objects is $L^\star$. Their separation L in the first frame is related by $L = L^\star/\gamma_u$ where $\gamma_u = 1/\sqrt{1 - u^2/c^2}$.

The velocity of the objects in the second frame is $u' = (u - v)/(1 - uv/c^2)$, according to the Lorentz transformation of velocity from the first frame. Then, we also have that $L' = L^\star/\gamma_{u'} = \gamma_u L/\gamma_{u'}$, where $\gamma_{u'} = 1/\sqrt{1 - u'^2/c^2}$, and that $\gamma_{u'} = \gamma\gamma_u(1 - u\, v/c^2)$.

If $u = 0$, then the first and third frames are the same, and $L' = L/\gamma = L^\star/\gamma$, which is the Lorentz contraction of distance L, which now corresponds to that, $L^\star$, between two objects at rest in the first frame.

[8] We could also have used the x-transformation of eq. (1),

$$x'_{i,0} = \gamma(x_{i,0} - v\, t_{i,0}) = \gamma\left(i\, L - v\frac{i\, v\, L}{c^2}\right) = \gamma\, i\, L\left(1 - \frac{v^2}{c^2}\right) = \frac{i\, L}{\gamma}\,, \tag{14}$$

or the time-transformation of eq. (2),

$$t = \gamma(t' + v\, x'/c^2), \qquad \frac{v\, x'_{i,0}}{c^2} = \frac{t_{i,0}}{\gamma} - t'_{i,0}\,, \qquad x'_{i,0} = \frac{c^2}{v}\frac{i\, v\, L}{\gamma\, c^2} = \frac{i\, L}{\gamma}\,. \tag{15}$$

[9] It is "amusing" that for $v > c/2$, the "final" position in the second frame for the object with $i = 0$, is at negative x', while its "final" position in the first frame is at positive x.

and in particular, recalling eq. (17),

$$v'_i(t'_{i,f}) = c = -v + a'_i t'_{i,f}, \qquad a'_i = \frac{c+v}{t'_{i,f}} = \frac{a}{\gamma}\frac{c+v}{c-v/2}\,. \tag{19}$$

Also, the position $x'_i(t')$ during this time interval would be related by,

$$x'_i(t') = x'_{i,0} - vt' + \frac{a'_i t'^2}{2} = \frac{i\,L}{\gamma} - vt' + \frac{a'_i t'^2}{2}, \tag{20}$$

and in particular,

$$x'_i(t'_{i,f}) = \frac{i\,L}{\gamma} + \frac{\gamma c}{a}\left(\frac{c}{2} - v\right) = \frac{i\,L}{\gamma} - v\frac{\gamma}{a}\left(c - \frac{v}{2}\right) + \frac{a'_i}{2}\frac{\gamma^2}{a^2}\left(c - \frac{v}{2}\right)^2 \qquad a'_i = \frac{a}{\gamma^3}\frac{c^2}{(c-v/2)^2}\,. \tag{21}$$

The two computations (19) and (21) of the supposedly constant acceleration a'_i disagree, which contradiction indicates that this acceleration is not actually constant.

Thus, we have a first lesson in the relativity of acceleration:
Acceleration that is constant in one inertial frame is not constant in another.[10]

We now examine the motion of the objects in the second frame in more detail, to determine the time dependence of the acceleration $a'_i(t')$.

2.2.2 Transformation of the History (x_i, t_i) from the First to the Second Frame

The history of object i in the first frame when accelerating is given in eq. (5) with $u = 0$,

$$x_i(t_i) = x_{i,0} + \frac{a}{2}(t_i - t_{i,0})^2 = i\,L + \frac{a}{2}\left(t_i - \frac{i\,v\,L}{c^2}\right)^2 \qquad \left(t_{i,0} = \frac{i\,v\,L}{c^2} < t_i < t_{i,0} + \frac{c}{a}\right). \tag{22}$$

The transformation of this history into the second frame can be made using eq. (1),

$$\frac{x'_i}{\gamma} = x_i - v\,t_i = i\,L + \frac{a}{2}\left(t_i - \frac{i\,v\,L}{c^2}\right)^2 - v\,t_i, \tag{23}$$

$$\frac{t'_i}{\gamma} = t_i - \frac{v\,x_i}{c^2} = t_i - \frac{v}{c^2}\left[i\,L + \frac{a}{2}\left(t_i - \frac{i\,v\,L}{c^2}\right)^2\right]. \tag{24}$$

To convert eq. (23) to a history of the form $x'_i(t'_i)$ we need to know t_i as a function of t'_i, which we can obtain from eq. (24),

$$\frac{a\,t_i^2}{2} - \frac{c^2 t_i}{v} - \frac{2i\,a\,v\,L\,t_i}{2c^2} + i\,L + \frac{i^2 a\,v^2 L^2}{2c^4} + \frac{c^2 t'_i}{\gamma\,v} = 0,$$

$$t_i^2 - 2\left(\frac{c^2}{a\,v} + \frac{i\,v\,L}{c^2}\right)t_i + \frac{2i\,L}{a} + \frac{i^2 a\,v^2 L^2}{c^4} + \frac{2c^2 t'_i}{\gamma\,a\,v} = 0, \tag{25}$$

$$t_i = \frac{c^2}{a\,v} + \frac{i\,v\,L}{c^2} \pm \sqrt{\left(\frac{c^2}{a\,v} + \frac{i\,v\,L}{c^2}\right)^2 - \left(\frac{2i\,L}{a} + \frac{i^2 v^2 L^2}{c^4} + \frac{2c^2 t'_i}{\gamma\,a\,v}\right)}. \tag{26}$$

[10] An exception is the case where the acceleration vector **a** is perpendicular to the relative velocity vector **v** between the two frames. Here, the acceleration is constant in both frames, although with values that differ by a factor of γ^2.

Note that in arriving at eq. (26), we have divided by the acceleration a in the first frame, so if this acceleration is zero, this relation is not valid.

Since $t_i = i\,v\,L/c^2$ when $t'_i = 0$, we use the negative square root,

$$t_i = \frac{i\,v\,L}{c^2} + \frac{c^2}{a\,v}\left(1 - \sqrt{1 - \frac{2a\,v\,t'_i}{\gamma\,c^2}}\right), \tag{27}$$

$$\left(t_i - \frac{i\,v\,L}{c^2}\right)^2 = \frac{2c^4}{a^2v^2}\left(1 - \sqrt{1 - \frac{2a\,v\,t'_i}{\gamma\,c^2}} - \frac{a\,v\,t'_i}{\gamma\,c^2}\right). \tag{28}$$

Using this in eq. (23), we obtain the history of object i in the second frame,

$$\begin{aligned}
\frac{x'_i}{\gamma} &= i\,L + \frac{c^4}{a\,v^2} - \frac{c^4}{a\,v^2}\sqrt{1 - \frac{2a\,v\,t'_i}{\gamma\,c^2}} - \frac{c^2t'_i}{\gamma\,v} - \frac{i\,v^2\,L}{c^2} - \frac{c^2}{a} + \frac{c^2}{a}\sqrt{1 - \frac{2a\,v\,t'_i}{\gamma\,c^2}} \\
&= \left(1 - \frac{v^2}{c^2}\right)\left(i\,L + \frac{c^4}{a\,v^2} - \frac{c^4}{a\,v^2}\sqrt{1 - \frac{2a\,v\,t'_i}{\gamma\,c^2}}\right) - \frac{c^2t'_i}{\gamma\,v}, \qquad (29) \\
x'_i(t'_i) &= \frac{1}{\gamma}\left(i\,L + \frac{c^4}{a\,v^2} - \frac{c^4}{a\,v^2}\sqrt{1 - \frac{2a\,v\,t'_i}{\gamma\,c^2}}\right) - \frac{c^2t'_i}{v}. \qquad (30)
\end{aligned}$$

Taking time derivatives of eq. (30), we obtain the velocity and acceleration of object i in the second frame,[11]

$$v'_i(t'_i) = \frac{c^2}{\gamma^2 v\sqrt{1 - \frac{2a\,v\,t'_i}{\gamma\,c^2}}} - \frac{c^2}{v}, \qquad v'_i(0) = -v, \qquad v'_i(t'_{i,f}) = c \quad \text{for} \quad t'_{i,f} = \frac{\gamma}{a}\left(c - \frac{v}{2}\right), \tag{31}$$

$$a'_i(t'_i) = \frac{a}{\gamma^3\left(1 - \frac{2a\,v\,t'_i}{\gamma\,c^2}\right)^{3/2}}. \tag{32}$$

The velocity v'_i of eq. (31), as observed in the second frame, begins at $-v$ at $t' = 0$ when the acceleration starts, as expected, and reaches the speed of light at time $t'_{i,f}$ as previously found in eq. (17). The acceleration a'_i is the same for all objects i as observed in the second frame, but is time dependent, while the acceleration a in the first frame is constant. This illustrates the "relativity of acceleration" as discussed in sec. 2.2.1 above.

2.2.3 No Collisions in the Second Frame

A consequence of eqs. (30) and (32) is that adjacent objects maintain constant separation L/γ during their acceleration as observed in the second frame, and hence do not collide, in agreement with the argument of sec. 2.1.1 above.

[11] With some effort we can rewrite eq. (32) as $a'_i = a/\gamma^3(1 - v_iv/c^2)^3$, which agrees with the expressions for a'_x found in eqs. (45) and (55), noting that the acceleration $\mathbf{a}_i = (a, 0, 0)$ and velocity $\mathbf{u}_i = (v_i, 0, 0)$ of object i in the first frame are both in the x-direction.

2.3 Separation of Neighboring Objects According to an Observer that Accelerates with One of Them

As discussed, for example, in Appendix B of [19], the separation between two objects in an accelerated frame of reference at some moment is that same as that in an inertial frame with the same instantaneous velocity as the accelerated frame (at that moment).

We consider objects 0 and -1 for the case that their initial velocity u is zero in the unprimed inertial frame, and emphasize the accelerated frame associated with object 0.[12] Then according to eqs. (4) and (5), their accelerated motion with respect to the unprimed frame is given by,

$$v_i(t) = a(t - t_{i,0}), \quad x_i(t) = iL + \frac{a}{2}(t - t_{i,0})^2 \qquad \left(t_{i,0} = \frac{i\,v\,L}{c^2} < t < t_{i,0} + \frac{a}{c}\right), \tag{33}$$

At some later time $0 < t < -vL/c^2 + a/c$, when both objects -1 and 0 are still accelerating their separation in the unprimed frame are,

$$x_0(t) - x_{-1}(t) = L - \frac{av^2L^2}{2c^4} - \frac{avL\,t}{c^2} > 0, \tag{34}$$

and object 0 has velocity $v_0(t) = a\,t$ with respect to the unprimed frame. As discussed in sec. 2.1.1 above, the two objects never collide before they stop accelerating (when their speeds reach c, although their separation (in the unprimed frame) decreases with time.

Qualitatively, we also expect that the separation between the two objects decreases with time according to an observer accelerated along with object zero.

However, a detail computation of this is complicated, because we must deduce to location of object -1 in the $''$ inertial frame that has velocity $v'' = v_0(t) = at$ with respect to the unprimed frame, at time t''_0 corresponding to time t in the unprimed frame.

The Lorentz transformation between these two inertial frames is,

$$x'' = \gamma''(x - v''\,t), \qquad t'' = \gamma''(t - v''\,x/c^2), \tag{35}$$

$$x = \gamma''(x'' + v''\,t''), \qquad t = \gamma''(t'' + v''\,x''/c^2), \tag{36}$$

$$\text{where} \qquad \gamma'' = \frac{1}{\sqrt{1 - (v''/c)^2}}\,. \tag{37}$$

At time t, object 0 has coordinates,

$$x''_0 = \gamma''\left(\frac{at^2}{2} - v''\,t\right), \qquad t''_0 = \gamma''\left(t - \frac{av''\,t^2}{c^2}\right). \tag{38}$$

We now need to deduce the position $X'' = x''_{-1}(t''_0)$ of object -1 in the $''$ frame at time t''_0, such that the separation Δ between objects -1 and 0 in the accelerated frame of object 0 is

$$\Delta = x''_0 - X''. \tag{39}$$

[12] We consider the second object to have index -1 rather than 1 so that it is already accelerating at the time $t = 0$ when object 0 begins accelerating with respect to the unprimed inertial frame.

For this, we use the Lorentz transformation (36) of X'' and t_0'' back into the unprimed frame, where when know the motion of object -1,

$$x_{-1} = \gamma''(X'' + v''\, t_0''), \qquad t_{-1} = \gamma''(t_0'' + v''\, X''/c^2), \tag{40}$$

These coordinates, x_1 and t_{-1} are related by eq. (33), which gives us a quadratic equation for the quantity X'',

$$x_{-1} = \gamma''(X'' + v''\, t_0''), = \frac{a}{2}\left[\gamma''\left(t_0'' + \frac{v''\, X''}{c^2}\right) + \frac{vL}{c^2}\right]^2 - L, \tag{41}$$

After solving this equation for X'', all the ingredients of the separation (39) are in hand, but an analytic expression for Δ is very cumbersome.

A Appendix: Transformations of Velocity and Acceleration

The transformation of velocity in special relativity was considered by Einstein in sec. 5 of his first paper [2], for a particle with velocity $\mathbf{u} = d\mathbf{x}/dt$ in the inertial lab frame, and velocity $\mathbf{u}'$ in an inertial frame that has (constant) velocity $\mathbf{v} = v\,\hat{\mathbf{x}}$ with respect to the lab frame.

The Lorentz transformation of coordinates (x, y, x, z) in the lab frame to coordinates (x', y', z', t') in the moving frame can be written as,[13]

$$t' = \gamma(t - vx/c^2), \qquad x' = \gamma(x - vt), \qquad y' = y, \qquad z' = z, \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}\,. \tag{42}$$

Taking time derivatives of eq. (42), we find,[14]

$$\frac{dt'}{dt} = \gamma(1 - vu_x/c^2), \qquad u_x' = \frac{dx'}{dt'} = \frac{dx'/dt}{dt'/dt} = \frac{u_x - v}{1 - vu_x/c^2}\,,$$
$$u_y' = \frac{dy'}{dt'} = \frac{dy/dt}{dt'/dt} = \frac{u_y}{\gamma(1 - vu_x/c^2)}\,, \qquad u_z' = \frac{dz'}{dt'} = \frac{dz/dt}{dt'/dt} = \frac{u_z}{\gamma(1 - vu_x/c^2)}\,. \tag{43}$$

Einstein [2] emphasized the magnitudes u and u', defining α to be the angle between $\mathbf{u}$ and $\mathbf{v}$, such that $u_x = u\cos\alpha$,

$$u' = \frac{\sqrt{(u\cos\alpha - v)^2 + (u\sin\alpha)^2(1 - v^2/c^2)}}{\gamma(1 - vu\cos\alpha/c^2)^3} = \frac{\sqrt{u^2 + v^2 - 2vu\cos\alpha - (vu\sin\alpha/c)^2}}{1 - vu\cos\alpha/c^2}\,. \tag{44}$$

[13] As Einstein discussed in secs. 1 and 2 of [2], the theory of special relativity differs from Galilean relativity by the presence of three effects: the relativity of simultaneity, the Lorentz contraction, and the time dilation between observations made in inertial frames that are in motion with respect to one another. In the Lorentz transformation (42), we can say that the factor of γ in the relation $t' = \gamma(t - vx/c^2)$ is associated with the time dilation, while the offset vx/c^2 is a manifestation of the relativity of simultaneity, and the factor of γ in the relation $x' = \gamma(x - vt)$ is associated with the Lorentz contraction.

[14] Following the sense of the preceding footnote, we can say that the factor $1 - vu_x/c^2$ which appears in eq. (43) is a result of the relativity of simultaneity, while the factor γ which appears in the equations for u_y' and u_z' is a result of the time dilation.

Taking time derivatives of eq. (43), we find,

$$
\begin{aligned}
a'_x &= \frac{du'_x}{dt'} = \frac{du'_x/dt}{dt'/dt} = \frac{a_x}{\gamma^3(1-vu_x/c^2)^3}\,, \\
a'_y &= \frac{du'_y}{dt'} = \frac{du'_y/dt}{dt'/dt} = \frac{a_y}{\gamma^2(1-vu_x/c^2)^2} + \frac{a_x v}{c^2}\frac{u_y}{\gamma^2(1-vu_x/c^2)^3}\,, \\
a'_z &= \frac{du'_z}{dt'} = \frac{du'_z/dt}{dt'/dt} = \frac{a_z}{\gamma^2(1-vu_x/c^2)^2} + \frac{a_x v}{c^2}\frac{u_z}{\gamma^2(1-vu_x/c^2)^3}\,.
\end{aligned} \tag{45}
$$

These results do not appear in Einstein's first paper [2],[15] where he considered only the limit that u, v and u' are all small compared to c, but ignored that $\gamma \approx 1$ in this limit, to write,

$$
a'_x \approx \frac{a_x}{\gamma^3}\,, \qquad a'_y \approx \frac{a_y}{\gamma^2}\,, \qquad a'_z \approx \frac{a_z}{\gamma^2}\,. \tag{46}
$$

Then, using the forms (46) in the lab-frame relation $\mathbf{F} = m\,\mathbf{a}$ he made the unfortunate inference that in the moving frame we could write $F'_x = \gamma^3 m\, a'_x$, $F'_y = \gamma^2 m\, a'_y$, and $F'z = \gamma^2 m\, a'_z$, which led him to introduce the notions of *longitudinal mass* $\gamma^3 m$ and *transverse mass* $\gamma^2 m$.[16]

A important clarification was made by Planck (1906) [21] who argued that Newton's 2$^{\rm nd}$ is better expressed as: force equals rate of change of momentum, and used the Lorentz force on an electric charge e and rest mass m as an example,

$$
\mathbf{F} = \frac{d\mathbf{P}}{dt} = \frac{d}{dt}(\gamma_u m\,\mathbf{u}) = \gamma_u m\left(\mathbf{a} + \frac{\gamma_u^2(\mathbf{a}\cdot\mathbf{u})}{c^2}\mathbf{u}\right) = e\left(\mathbf{E} + \frac{\mathbf{u}}{c}\times\mathbf{B}\right), \quad \gamma_u = \frac{1}{\sqrt{1-u^2/c^2}}\,. \tag{47}
$$

In this, one could speak of a single relativistic mass $\gamma_u m$, although Einstein never liked this terminology.[17]

The next step in the evolution of the concepts of velocity and acceleration in special relativity was the introduction of 4-vectors by Minkowski (1908) [25]. After Minkowski's untimely death in Jan. 1909, the use of 4-vectors was quickly championed by Sommerfeld, including [26] on the velocity 4-vector.

We consider the position 4-vector $x_\mu = (x_0, x_1, x_2, x_3) = (ct, x, y, z)$, and the velocity 4-vector u_μ,

$$
u_\mu = \frac{dx_\mu}{d\tau} = (\gamma_u c, \gamma_u \mathbf{u}), \qquad \gamma_u = \frac{1}{\sqrt{1-u^2/c^2}}\,, \tag{48}
$$

where $\mathbf{u} = d\mathbf{x}/dt$ in the (inertial) "lab" frame in which the components of u_μ are given above, and $d\tau = dt/\gamma_u$ is the interval of proper time for an observer/clock with velocity $\mathbf{u}$. We can

[15] Equation (45) may have first appeared in sec. 42, p. 48 of [20].

[16] Einstein was aware of the doubtful character of his inference, commenting [2]: *With a different definition of force and acceleration we should naturally obtain other values for the masses. This shows us that in comparing different theories of the motion of the electron we must proceed very cautiously.*

[17] The convenience of introducing the relation $m = \gamma_u m_0$, where m_0 in the invariant/rest mass was emphasized by Lewis, starting with eq. (15), p. 711 of [22] (1908). The term relativistic mass was perhaps first used by Born, eq. (87), p. 203 of [23] (1922). A review of Einstein's attitudes on this is given in [24].

transform eq. (48) to the $'$ inertial frame that has velocity $\mathbf{v}$ with respect to the "lab" frame by first decomposing velocity $\mathbf{u}$ into components $\mathbf{u}_\parallel = (\mathbf{u}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}}$ and $\mathbf{u}_\perp = \mathbf{u} - \mathbf{u}_\parallel$ that are parallel and perpendicular to the velocity $\mathbf{v}$. Then, the Lorentz transformation $(\gamma_{u'}c, \gamma_{u'}\mathbf{u}')$ of the 4-vector $(\gamma_u c, \gamma_u \mathbf{u})$ has components,

$$\gamma_{u'}c = \gamma(\gamma_u c - \gamma_u u_\parallel v/c), \qquad \gamma_{u'}\mathbf{u}'_\parallel = \gamma(\gamma_u \mathbf{u}_\parallel - \gamma_u \mathbf{v}), \qquad \gamma_{u'}\mathbf{u}'_\perp = \gamma_u \mathbf{u}_\perp, \tag{49}$$

$$\mathbf{u}'_\parallel = \frac{\mathbf{u}_\parallel - \mathbf{v}}{1 - u_\parallel v/c^2} = \frac{\mathbf{u}_\parallel - \mathbf{v}}{1 - \mathbf{u}\cdot\mathbf{v}/c^2}\,, \qquad \mathbf{u}'_\perp = \frac{\mathbf{u}_\perp}{\gamma(1 - \mathbf{u}\cdot\mathbf{v}/c^2)}\,, \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}\,, \tag{50}$$

as previously found in eq. (43).

It is now natural to consider the 4-acceleration a_μ to be defined as,

$$a_\mu = \frac{du_\mu}{d\tau} = \left(\gamma_u c \frac{d\gamma_u}{dt}, \gamma_u \frac{d\ \gamma_u \mathbf{u}}{dt}\right) = \left(\gamma_u^4 (\mathbf{a}\cdot\mathbf{u})/c, \gamma_u^2 \mathbf{a} + \gamma_u^4 (\mathbf{a}\cdot\mathbf{u})\mathbf{u}/c^2\right), \tag{51}$$

where $\mathbf{a} = d\mathbf{u}/dt = \mathbf{a}_\parallel + \mathbf{a}_\perp$. The Lorentz transformation of eq. (51) has components,

$$\begin{aligned}
\gamma_{u'}^4 (\mathbf{a}'\cdot\mathbf{u}')/c &= \gamma\left[\gamma_u^4 (\mathbf{a}\cdot\mathbf{u})/c - \gamma_u^2 a_\parallel v/c - \gamma_u^4 (\mathbf{a}\cdot\mathbf{u}) u_\parallel v/c^3\right] \\
&= \gamma\left[(1 - \mathbf{u}\cdot\mathbf{v}/c^2)\gamma_u^4 (\mathbf{a}\cdot\mathbf{u})/c - \gamma_u^2 (\mathbf{a}\cdot\mathbf{v})/c\right], \qquad (52) \\
\gamma_{u'}^2 \mathbf{a}'_\parallel + \gamma_{u'}^4 (\mathbf{a}'\cdot\mathbf{u}')\mathbf{u}'_\parallel/c^2 &= \gamma\left[\gamma_u^2 \mathbf{a}_\parallel + \gamma_u^4 (\mathbf{a}\cdot\mathbf{u})\mathbf{u}_\parallel/c^2 - \gamma_u^4 (\mathbf{a}\cdot\mathbf{u})\mathbf{v}/c^2\right], \qquad (53) \\
\gamma_{u'}^2 \mathbf{a}'_\perp + \gamma_{u'}^4 (\mathbf{a}'\cdot\mathbf{u}')\mathbf{u}'_\perp/c^2 &= \gamma_u^2 \mathbf{a}_\perp + \gamma_u^4 (\mathbf{a}\cdot\mathbf{u})\mathbf{u}_\perp/c^2, \qquad (54)
\end{aligned}$$

$$\mathbf{a}'_\parallel = \frac{\mathbf{a}_\parallel}{\gamma^3(1 - \mathbf{u}\cdot\mathbf{v}/c^2)^3}\,, \qquad \mathbf{a}'_\perp = \frac{\mathbf{a}_\perp}{\gamma^2(1 - \mathbf{u}\cdot\mathbf{v}/c^2)^2} + \frac{\mathbf{a}\cdot\mathbf{v}/c^2}{\gamma^2(1 - \mathbf{u}\cdot\mathbf{v}/c^2)^3}\mathbf{u}_\perp, \tag{55}$$

as previously found in eq. (45).[18,19]

A.1 $\mathbf{u} = \mathbf{v}$ (Mar. 6, 2022)

A special case of interest is for $\mathbf{u} = \mathbf{v}$ ($\mathbf{u}_\perp = 0$), in which case the $'$ frame is the instantaneous inertial rest frame of the accelerated particle. Then, $(1 - \mathbf{u}\cdot\mathbf{v}/c^2)^3 = 1/\gamma^6$, recalling that $\gamma = 1/\sqrt{1 - v^2/c^2}$, and the transformation of the acceleration is,

$$\mathbf{a}'_\parallel = \gamma^3 \mathbf{a}_\parallel, \qquad \mathbf{a}'_\perp = \gamma^2 \mathbf{a}_\perp, \qquad (\mathbf{u} = \mathbf{v}). \tag{56}$$

In particular, for acceleration along a straight line ($\mathbf{a}_\perp = 0$),

$$\mathbf{a}' = \gamma^3 \mathbf{a} \qquad (\mathbf{u} = \mathbf{v},\ \mathbf{a} \parallel \mathbf{v}). \tag{57}$$

This has the implication, according to the equivalence principle, that the apparent weight ma' of a particle of mass m which has acceleration $\mathbf{a}$ with respect to the inertial lab frame is $\gamma^3 ma$ for acceleration in a straight line. If the "particle" were an observer who noted a variation of (her)/his apparent weight with time, (s)he would know (s)he is not at rest in

[18] The author finds it surprising that the relations (55) seem to be incorrectly stated in eqs. (18)-(19), p. 318 of [27], and also in sec. 26 of [28] where eq. (193) is valid but eq. (194) is not.

[19] The transformation of acceleration is expressed in an alternative notation in sec. V of [29].

a gravitational field.[20] Of course, such variation would only be noticeable on the velocity became "large".

For acceleration a that is constant in the inertial lab frame, $v = at$ if the particle starts from rest at $t = 0$. Then, the particle would reach the speed of light at time $t = c/a$. At this time, the acceleration (57) in the inertial rest frame of the particle would be infinite, requiring an infinite force, and infinite expenditure of energy in that frame.[21]

A.2 Accelerated Frames of Reference (Apr. 29, 2022)

The preceding discussion considered only inertial frames. Some discussion of accelerated frames in special relativity is given in [30], including the fact the acceleration of one frame with respect to another accelerated frame is not necessarily equal and opposite to the acceleration of the second accelerated frame with respect to the first. This contrasts with the fact that the relative velocity one inertial frame with respect to another is equal and opposite to that of the second inertial frame with respect to the first.

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