

Capacitor-Driven Railgun: Magnetic Fields Doing Work

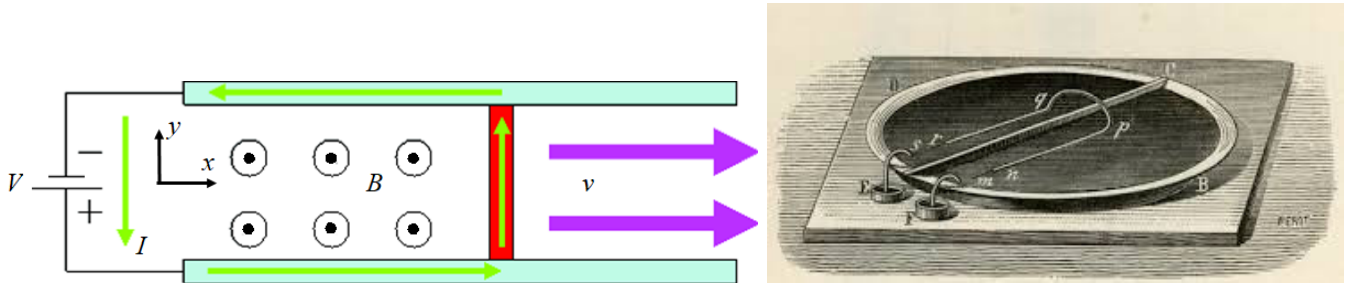
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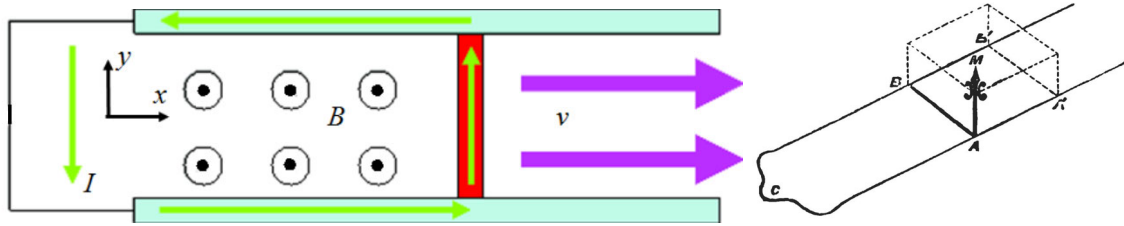
1 Problem

Discuss force, work and energy in a capacitor-drive railgun, as illustrated on the left below (from [263]),^{1,2} and in the original conception due to Ampère and de La Rive (1822) [39] as illustrated on the right below.³



Examples like these are used to illustrate the claim that “magnetic forces do no work” [161, 165, 270, 271, 280], which implies that some other force(s) than magnetism are “at work” here.

Consider also the variant shown below⁴ in which the railgun circuit is initially superconducting with current I_0 and fixed (superconducting) crosspiece), but at time $t = 0$ the crosspiece goes normal, takes on electrical resistance R , and is set free to move.⁵



¹As this note has an historical flavor, it seemed appropriate that the references be listed in chronological order, by date of submission for publication.

²The configuration on the left was first discussed by Davy in 1820 [34], with an external magnetic field.

³The right figure is from p. 331 of [99]. A railgun was patented by Birkeland in 1902 [109, 219]. For another contribution of Birkeland, see [279]. The literature on railguns is now vast; a sample is [164, 252, 272, 281].

⁴A railgun was discussed by Maxwell in Arts. 594-596 of [90], as sketched on in the figure on right.

⁵A simple railgun variant is now featured at sites such as

<http://sci-toys.com/scitoys/scitoys/electro/railgun/railgun.html>

In this, the magnetic field that provides the Lorentz force is due to strong permanent magnets that form the wheels of a rolling crosspiece. The direction of the axial field in the two wheels must be opposite (such that a single rolling magnet would not suffice).

This configuration has much in common with Faraday’s homopolar generator of 1832; see sec. A.16.4 below.

2 Solution

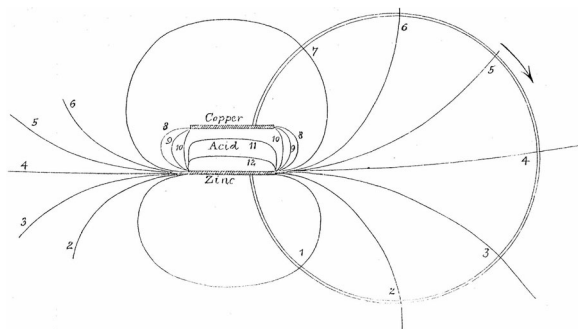
This problem is not static,⁶ so that it is not clear that methods of magnetostatics will suffice to analyze it. Yet, the statement “magnetic forces do no work” is invoked in the context of magnetostatics [271], where electricity and magnetism are not yet unified into a dynamical theory of electromagnetic field, as per Maxwell [85]. However, in electrostatics, and in the absence of a magnetic field, electric forces also do no work.

As noted by Poynting [97], in the case of work done by electromagnetic fields, \mathbf{E} and \mathbf{B} , the flux of energy in the field is described by the vector field,

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B}, \quad (1)$$

in Gaussian units, where c is the speed of light in vacuum. Whenever electromagnetic work is done, with a consequent flow of electromagnetic energy, both electric and magnetic fields must be present. That is, magnetic forces/fields, by themselves, and electric forces/fields, by themselves, do no work.

The figure below, from Poynting [97], illustrates lines of \mathbf{S} flowing from a battery to points along a circular resistor.



We find below that for the examples of railguns with no “battery”, nonzero work is done on the crosspieces. While the capacitor-driven railgun has a nonzero initial electric field inside the capacitor, the initially superconducting railgun has only an initial magnetic field. In both cases, the initial stored electromagnetic field energy is converted into final kinetic energy plus Joule heating. The final kinetic energy of the crosspiece equals the work done by the component of the magnetic force perpendicular to that crosspiece, such that in the view of this author, it is best to say that railguns are examples where magnetic forces do work.⁷

2.1 Ampère *vs.* Biot-Savart-Grassmann

Despite the need for a dynamical view of a railgun, it is natural to attempt a solution using static methods as much as possible.

⁶In another nonstatic experiment, Ampère and de La Rive came close to discovering electromagnetic induction [106, 194].

⁷In railguns the magnetic fields are due to conduction currents. In examples with permanent magnets, whose fields are associated with the permanent magnetic moment of electrons, the magnetic field “clearly” does work on the permanent magnetization. For one such example, see [262]. Consider also a refrigerator magnet being pulled out of your hand as it nears the surface of the fridge.

In static examples, the notion of fields that propagate with a finite speed is not necessary, and methods based on instantaneous action at a distance are sufficient. Indeed, the first great analysis of static magnetic phenomena involving electric currents, that of Ampère [51], did not involve the concept of a magnetic field (or an electric field, and did not associate electric current with moving charges). Similarly, the extension of Ampère’s analysis by Weber [104] to include nonstatic examples involving moving electric charges was also an instantaneous action-at-a-distance theory, although the velocities of charges were scaled by a velocity C deduced from considerations of electrical and magnetic units, whose numerical value is close to $\sqrt{2}$ times the speed c of light.⁸ The railgun problem is occasionally analyzed via Weber’s electrodynamics [123, 244], but we will not pursue this here.

Contemporary discussions of magnetic forces tend to assume as valid the Lorentz force law⁹ for an electric charge q with velocity \mathbf{v} in external electric field \mathbf{E} and magnetic field \mathbf{B} ,

$$\mathbf{F}_q = q \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right), \quad (3)$$

in Gaussian units, with c being the speed of light in vacuum. Applying this law to electrical

⁸In 1846, p. 144 of [104], Weber wrote the force between electric charges e and e' , separated by distance r , is directed along their line of centers, with magnitude,

$$F = \frac{ee'}{r^2} \left[1 - a^2 \left(\frac{dr}{dt} \right)^2 + 2a^2 r \frac{d^2r}{dt^2} \right]. \quad (2)$$

This was the first published force law for moving charges (which topic Ampère refused to speculate upon). The constant a has dimensions of velocity⁻¹, and was later (1856) written by Kohlrausch and Weber [79] as $1/C$, who noted that their C is the ratio of the magnetic units to electrical units in the description of static phenomenon, which they determined experimentally to have a value close to 4.4×10^8 m/s. Apparently, they regarded it as a coincidence that their C is roughly $\sqrt{2}$ times the speed c of light.

For an extensive discussion of Weber’s electrodynamics, see [240]. Maxwell gave a review of the German school of electrodynamics of the mid 19th century in the final chapter 23, part 4, of his *Treatise* [90].

⁹Maxwell discussed a version of the railgun in Arts. 594-598 of [90], and then in Arts. 598-599 considered the “electromotive intensity” to be eq. (3) divided by q . However, he seems not to have made the inference that eq. (3) represents the force on a moving charge, as pointed out by FitzGerald [96].

The force law eq. (3) perhaps first explicitly stated by Heaviside (1889), eq. (10) of [100], although Maxwell wrote it in a somewhat disguised form on p. 342 of [83] (1861), and in Art. 599 of [90]. Like Heaviside, Lorentz (1892) gave the force law in the form $q(\mathbf{D} + \mathbf{v}/c \times \mathbf{H})$, eq. (113) of [102]. The debate as to whether the force depends on \mathbf{B} or \mathbf{H} was settled experimentally in favor of \mathbf{B} only in 1944 [132].

In contemporary usage, as for Maxwell, the velocity \mathbf{v} in the Lorentz force law is that of the charge in the (inertial) lab frame where \mathbf{F} , \mathbf{E} and \mathbf{B} are measured. However, in Lorentz’ original view the velocity was to be measured with respect to the supposed rest frame of the ether. See, for example, [237].

circuits 1 and 2, the force on circuit 1 due to circuit 2 is the so-called Biot-Savart law,¹⁰

$$\begin{aligned}\mathbf{F}_{\text{on } 1} &= \oint_1 d\mathbf{F}_{\text{on } 1} = \oint_1 \frac{I_1 d\mathbf{l}_1 \times \mathbf{B}_2 \text{ at } 1}{c} = \oint_1 \oint_2 d^2\mathbf{F}_{\text{on } 1}, \quad \mathbf{B}_2 \text{ at } 1 = \oint_2 \frac{I_2 d\mathbf{l}_2 \times \hat{\mathbf{r}}}{cr^2}, \\ d^2\mathbf{F}_{\text{on } 1} &= I_1 d\mathbf{l}_1 \times \frac{I_2 d\mathbf{l}_2 \times \hat{\mathbf{r}}}{c^2 r^2} = I_1 I_2 \frac{(\hat{\mathbf{r}} \cdot d\mathbf{l}_1) d\mathbf{l}_2 - (d\mathbf{l}_1 \cdot d\mathbf{l}_2) \hat{\mathbf{r}}}{c^2 r^2} \neq -d^2\mathbf{F}_{\text{on } 2}\end{aligned}\quad (4)$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ is the distance from a current element $I_2 d\mathbf{l}_2$ at \mathbf{r}_2 to element $I_1 d\mathbf{l}_1$ at \mathbf{r}_1 .¹¹ Ampère held a rather different view [51], that,

$$\mathbf{F}_{\text{on } 1} = \oint_1 \oint_2 d^2\mathbf{F}_{\text{on } 1}, \quad d^2\mathbf{F}_{\text{on } 1} = I_1 I_2 [3(\hat{\mathbf{r}} \cdot d\mathbf{l}_1)(\hat{\mathbf{r}} \cdot d\mathbf{l}_2) - 2 d\mathbf{l}_1 \cdot d\mathbf{l}_2] \frac{\hat{\mathbf{r}}}{c^2 r^2} = -d^2\mathbf{F}_{\text{on } 2}. \quad (5)$$

Ampère considered that the laws of electrodynamics should respect Newton’s third law, of action and reaction, whereas the Biot-Savart/Lorentz law (4) does not. This “minor” detail is seldom discussed in textbooks,¹² and it held up acceptance of eq. (4) in preference to eq. (5) for about 70 years, 1820-1890.¹³

In 1822, Ampère (with de La Rive) performed the experiment sketched in the right figure of p. 1, the results of which Ampère interpreted as vindicating his force law (5).

Maxwell reviewed Ampère’s experiment in Art. 687 of [90], arguing that since the forms (4) and (5) give the same result for the force on a closed circuit, the experiment could not distinguish between these two forms.

However, the issues here are extremely delicate, since Ampère’s “hairpin” experiment was not about the force of one closed circuit on another, but rather concerned the force on one

¹⁰Biot and Savart [21, 28, 46] had no concept of the magnetic field \mathbf{B} of an electric current I , and discussed only the force on a magnetic pole p , as $p \oint I d\mathbf{l} \times \hat{\mathbf{r}}/cr^2$, although not, of course, in vector form. The form (4) can be traced to Ampère (1825) [49], and Grassmann (1845) [62], still not in vector form. The vector relation $\mathbf{F}_{\text{on } 1} = \oint_1 I_1 d\mathbf{l}_1 \times \mathbf{B}_2 \text{ at } 1/c$ appears without attribution as eq. (11) of Art. 603 of Maxwell’s *Treatise* [90], while Einstein may have been the first to call this the Biot-Savart law, in sec. 2 of [112].

Heaviside, p. 551 and 559 of [105], discussed the form $d\mathbf{F} = \rho\mathbf{E} + \mathbf{\Gamma} \times \mathbf{H}$, where ρ is the electric charge density and $\mathbf{\Gamma} = \nabla \times \mathbf{H} = \mathbf{J} + \partial\mathbf{D}/\partial t$, where \mathbf{J} is the conduction current density \mathbf{J} and $\partial\mathbf{D}/\partial t$ is the “displacement current” density. However, the present view is that the “displacement current” does not experience a magnetic force.

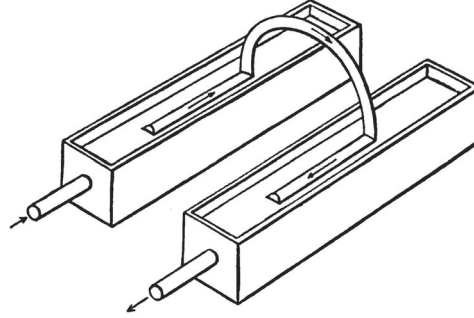
The earliest description in English of eq. (4) as the Biot-Savart law may be in sec. 7-6 of [145].

¹¹If we follow Ampère in defining a “current element” as being electrically neutral, which is a good (but not exact [269]) approximation for currents in electrical circuits, then an isolated, moving charge is not a “current element” (contrary to remarks such as in [226]). A wire that is used to discharge a capacitor could be considered as an example of an Ampèrian current element when it carries the transient current. The magnetic forces on a pair of such current elements would not obey Newton’s third law, but overall momentum conservation is observed when one takes into account the momentum stored in the electromagnetic fields of the system. This is easier to analyze for a pair of moving charges [134] than for a pair of discharging capacitors.

¹²One exception is sec. 7-5 of [145].

¹³Maxwell gave an intricate discussion in Arts. 502-526 of his *Treatise* [90], in which he pointed out that experiments on the forces between closed circuits cannot fully determine an expression for the magnetostatic forces, and that one arbitrary assumption is required to arrive at a “law”. He considered (Art. 526) four such assumptions, including Ampère’s that the force law obey Newton’s third law, and Grassmann’s that the force is zero between collinear current elements; Maxwell then expressed his preference of Ampère’s form, although in Art. 599 he displayed the Lorentz force law without comment as to its relation to the forms of Ampère and Grassmann.

part of a single closed circuit due to the rest of that circuit. Namely, the experiment involved a U-shaped wire floating on two pools of mercury, and the floating wire was observed to move away from the fixed battery at the left of the figure below (from Art. 687 of [90]), but not shown.



According to the Biot-Savart law, eq. (4), the magnetic force on a current element is perpendicular to the current, while in Ampère's formulation, eq. (5), the force appears to have a component along the current. In particular, for the case of collinear current elements, the Biot-Savart force between them is zero, while Ampère's force is nonzero and along the common direction of the currents. Ampère argued that the observed repulsion of one part of the circuit (shown above) from the other confirmed the validity of his force law (5).

Ampère's purported longitudinal force between collinear current elements has come to be called the **Ampère tension**, and was the subject of considerable controversy in the 20th century. The first advocate of the Ampère tension was Hering [113, 116, 117, 118, 119, 120, 121], whose concerns inspired several sympathetic comments, [124, 126, 127, 129, 130, 133, 135, 136, 138, 142, 143, 144, 159]. Hering's notion of a tension along conductors was motivated in part by variants of Ampère's "hairpin" experiment, as shown below [116].

FIG. 4.

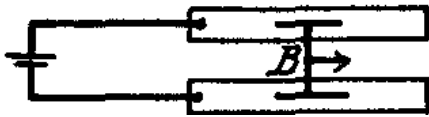


FIG. 5.

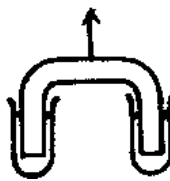


FIG. 11.

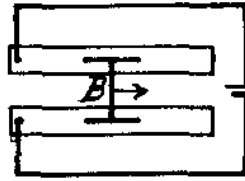
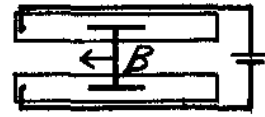


FIG. 12.



In the 1960s, a more strident advocate of Ampère tension emerged, Graneau [156, 171, 172, 173, 175, 177, 180, 181, 185, 188, 189, 196, 197, 198, 199, 218, 255],¹⁴ who also argued that the phenomenon of "exploding wires" is due to the supposed Ampère tension associated with collinear current elements, which might be large for extreme pulses of electric current [188, 198, 222, 250].¹⁵ Response to Graneau's claims was generally skeptical [174, 179, 183, 186, 187, 192, 195, 201, 202, 207, 209, 210, 211, 213, 214, 215, 216, 217, 223, 224, 225, 226, 238, 241, 245, 246, 248, 251, 259, 264, 273], but a few supporters are on record

¹⁴Graneau was an advocate of several nonstandard physics views,
<http://www.infinite-energy.com/iemagazine/issue114/graneau.html>

¹⁵Large current pulses lead to large Joule heating, which leads to transient thermal expansion in all directions, including that along the wire. The standard view is that this impulsive thermal expansion is the cause of exploding wires, as discussed in [195]. That is, the phenomenon of exploding wires can be attributed to "Joule tension" rather than to "Ampère tension".

[176, 182, 193, 184, 206, 220, 221, 228, 230, 232, 234, 236, 258, 261, 275].

2.2 Equivalence of the Ampère and Biot-Savart Force Laws for Closed Circuits

According to the Biot-Savart form (4), the force on circuit element $I_1 d\mathbf{l}_1$ due to current I_2 in circuit 2 is,

$$d\mathbf{F}_{\text{on } d\mathbf{l}_1}^{(\text{B-S})} = I_1 d\mathbf{l}_1 \times \oint_2 \frac{I_2 d\mathbf{l}_2 \times \mathbf{r}}{cr^3} = \frac{I_1 I_2}{c^2} \oint_2 \frac{(\mathbf{r} \cdot d\mathbf{l}_1) d\mathbf{l}_2 - (d\mathbf{l}_1 \cdot d\mathbf{l}_2) \mathbf{r}}{r^3}. \quad (6)$$

To compare this with Ampère's form (5), it is useful to note the relations (given by Ampère),

$$d\mathbf{l}_2 = -\frac{\partial \mathbf{r}}{\partial l_2} dl_2, \quad \mathbf{r} \cdot d\mathbf{l}_2 = \mathbf{r} \cdot \frac{\partial \mathbf{r}}{\partial l_2} dl_2 = -r \frac{\partial r}{\partial l_2} dl_2. \quad (7)$$

Then,

$$\begin{aligned} d\mathbf{F}_{\text{on } d\mathbf{l}_1}^{(\text{A})} &= \frac{I_1 I_2}{c^2} \oint_2 \frac{3(\mathbf{r} \cdot d\mathbf{l}_1)(\mathbf{r} \cdot d\mathbf{l}_2)\mathbf{r}}{r^5} - \frac{2(d\mathbf{l}_1 \cdot d\mathbf{l}_2)\mathbf{r}}{r^3} \\ &= \frac{I_1 I_2}{c^2} \oint_2 \left(-\frac{3d\mathbf{l}_1}{r^4} \cdot \mathbf{r} \frac{\partial r}{\partial l_2} dl_2 \right) \mathbf{r} - \frac{2I_1 I_2}{c^2} \oint_2 \frac{(d\mathbf{l}_1 \cdot d\mathbf{l}_2)\mathbf{r}}{r^3}. \end{aligned} \quad (8)$$

The i th component of the first integral on the second line of eq. (8) can be written as,

$$\begin{aligned} -\oint_2 \frac{3d\mathbf{l}_{1,j} r_i r_j}{r^4} \frac{\partial r}{\partial l_2} dl_2 &= \oint_2 d\mathbf{l}_{1,j} \left[\frac{\partial}{\partial l_2} \left(\frac{r_i r_j}{r^3} \right) - \frac{r_i}{r^3} \frac{\partial r_j}{\partial l_2} - \frac{r_j}{r^3} \frac{\partial r_i}{\partial l_2} \right] dl_2 \\ &= \oint_2 \frac{(d\mathbf{l}_1 \cdot d\mathbf{l}_2) r_i}{r^3} + \oint_2 \frac{(d\mathbf{l}_1 \cdot \mathbf{r}) d\mathbf{l}_{2,i}}{r^3}. \end{aligned} \quad (9)$$

Using this in eq. (8), we have that,

$$d\mathbf{F}_{\text{on } d\mathbf{l}_1}^{(\text{A})} = \frac{I_1 I_2}{c^2} \oint_2 \frac{(\mathbf{r} \cdot d\mathbf{l}_1) d\mathbf{l}_2 - (d\mathbf{l}_1 \cdot d\mathbf{l}_2) \mathbf{r}}{r^3} = d\mathbf{F}_{\text{on } d\mathbf{l}_1}^{(\text{B-S})} \quad (10)$$

Considering circuit element $I_1 d\mathbf{l}_1$ to be part of circuit 1, distinct from circuit 2, we find,

$$\mathbf{F}_{\text{on } 1}^{(\text{A})} = \oint_1 d\mathbf{F}_{\text{on } d\mathbf{l}_1}^{(\text{A})} = \oint_1 d\mathbf{F}_{\text{on } d\mathbf{l}_1}^{(\text{B-S})} = \mathbf{F}_{\text{on } 1}^{(\text{B-S})} \quad (11)$$

Finally, since Ampère's force between a pair of circuit elements is along their line of centers,

$$\mathbf{F}_{\text{on } 1}^{(\text{A})} = -\mathbf{F}_{\text{on } 2}^{(\text{A})} = \mathbf{F}_{\text{on } 1}^{(\text{B-S})} = -\mathbf{F}_{\text{on } 2}^{(\text{B-S})}. \quad (12)$$

When either the Biot-Savart form (4) or the Ampère form (5) is applied to a pair of circuits, the total forces on the circuits are the same, and Newton's third law is satisfied.^{16,17}

¹⁶A derivation something like the above was first given by Ampère in 1825 [49], followed by Neumann in 1845 [63, 69, 156], and in more detail by Stefan in 1869 [87, 93, 217].

¹⁷A tacit assumption here is that effects of wave propagation can be ignored. For an example in which a pair of circuits emit radiation, with a resulting propulsive force on the circuits, see [282].

When considering the force of a single circuit on itself, one can worry that the integrals in eqs. (6) and (8) might diverge, such that the self force might not be zero. Since Ampère’s force law between pairs of current elements obeys Newton’s third law, one has confidence that this leads to $\mathbf{F}_{\text{self}}^{(A)} = -\mathbf{F}_{\text{self}}^{(A)}$ such that the self force is zero, but the case for the Biot-Savart form cannot be argued so quickly. Stefan [87] considered that physical circuits have wires of finite diameter, for which it is convincing that the equivalence of the Ampère and the Biot-Savart force laws for closed, filamentary circuits implies that the self force is also zero for the latter form.¹⁸

A corollary to the above argument is that the total force on a current element due to currents in closed circuits is perpendicular to the current element, according to the (static) force laws of both Ampère and Biot-Savart.

An extrapolation of the static force laws of both Ampère and of Biot-Savart-Grassmann is that if a circuit moves in response to the (initially static) magnetic force on it, then that magnetic force does work on the moving circuit. This extrapolation presumes that the static force laws are still approximately correct for examples where the motions have low velocity.¹⁹

2.3 Ampère’s Hairpin Experiment

In Ampère’s hairpin experiment, there is a small repulsive force between the opposite currents in the battery segment and crosspiece (for example, in Fig. 4 of [116], p. 4 above). However, the variant shown in Hering’s Fig. 12 shows that this force is not the dominant propulsive force on the crosspiece.

Ampère’s argument [39] was that his force law (5) indicates a repulsive force (the Ampère tension) between collinear current elements.

However, the equivalence between the force laws of Ampère and of Biot-Savart, discussed in sec. 2.2, indicates that the force on a current element due to the rest of the circuit is the same according to both laws, eq. (10), and perpendicular to the current element (recalling eq. (6)), such that Ampère’s explanation is not valid.²⁰

A force perpendicular to the crosspiece in the hairpin circuit can only be due to the force on the current in the crosspiece, and not on the currents in the portions of the moving hairpin that are parallel to the troughs of mercury. Since the magnetic force falls off as $1/r^2$, the strongest force on the crosspiece is due to the currents in the just mentioned portions

¹⁸This was argued (rather briefly) by Maxwell in Art. 687 of [90], and variants have been given in [91, 94, 114, 125, 128, 153, 156, 186, 187, 201, 202, 207, 208, 214, 215, 217, 223, 224, 225, 238, 241, 245, 246, 273].

Graneau initially [156] seemed to endorse the equivalence of the force laws of Ampère and of Biot-Savart, but in [175] he claimed that numerical computation “proved” that the self force on a circuit is nonzero according to the Biot-Savart law.

¹⁹As mentioned in footnote 6, Weber was the first to consider a force law for moving charges, but this involved instantaneous action at a distance. Effects of retardation, due to the finite speed of propagation of electromagnetic waves, on the electric and magnetic fields of a point electric charge were first considered by Lienard [107] and by Wiechert [108]. The computation of the retarded fields for electric charge and current distributions is reviewed in [247].

²⁰As remarked by Maxwell in Art. 687 of [90]:

It is manifest that since the formulae both of Ampère and of Grassmann give the same results for closed circuits, and since we have in the experiment only a closed circuit, no result of the experiment can favour one more than the other of these theories.

of the moving hairpin that are parallel to the troughs of mercury in Figs. 4 and 11 of [116], while in Fig. 12 the forces are due to the currents in the wires just outside the troughs and to the right of the cross piece. In particular, the Biot-Savart law is more direct than Ampère’s version in computing these forces.

A disconcerting feature remains that in the variants of Hering’s Figs. 4 and 11 [116], the force on the current in the crosspiece of the hairpin is due to the currents in the rest of the hairpin.²¹ This goes against a notion from mechanics that a rigid body cannot exert a net force on itself.

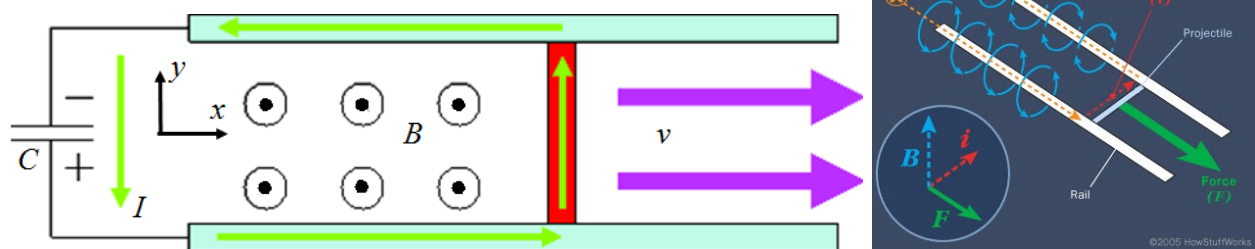
However, the U-shaped “hairpin” is not strictly a rigid body in that a current enters one end and leaves the other, and the force on the “rigid” part of the hairpin is due to the current which flows through it.²² The rigid part of the hairpin does not exert a force on itself, but rather a force back on the current, as will be discussed further in sec. 2.4.4. Hence, it seems to this author that concerns about a nonzero self force on the “rigid” hairpin are ill founded.

It remains to discuss force, work and energy for Ampère’s hairpin experiment. However, we content ourselves with such discussion for the closely related example of a railgun.

2.4 Railgun

2.4.1 Capacitor-Driven Railgun

Turning now to the railgun illustrated below, we suppose that the energy source is a capacitor bank of capacitance C , with initial electric charge Q on its plates. When this capacitor bank is switched into series with the rails, transient current $I(t)$ flows in the circuit, which generates a magnetic field that has a component in the z -direction (out of the page in the left figure below) when the current is in the $+y$ direction, such that the $\mathbf{J}/c \times \mathbf{B}$ force on the moving crosspiece is in the $+x$ direction.²³



We take the capacitor C to be at $x = 0$ and the crosspiece (in red) at $x = x_c \gg b$, where the various wires of radius $a \ll b$ lie in the plane $z = 0$, with x -segments centered on $y = 0$ and b . The magnetic field in the plane $z = 0$, between the x -segments and away from their

²¹This feature may have been first discussed in [126].

²²This comment appears in sec. II of [165].

²³In most railguns, the crosspiece slides between the rails. There exists a small literature on railguns in which the crosspiece rolls on the rails [80, 81, 82, 148, 160, 205, 227, 235, 239, 242, 249, 253, 254], where it is not obvious that the observed motion is due to the Lorentz force. Feb. 27, 2025. In 1820, Davy [34] demonstrated a precursor of a railgun in which a wire placed across two conductors connected to a battery would roll along the conductors when a magnetic field was directed perpendicular to the plane of the two conductors; this action is due to the $\mathbf{J} \times \mathbf{B}$ force on the current in the rolling wire.

ends, is approximately that associated with infinite-length wires,

$$B_z(b \ll x \ll x_c - b, a < y < b - a, 0) \approx \frac{2I}{cy} + \frac{2I}{c(b-y)}. \quad (13)$$

At the crosspiece, where $x = x_c$, the z -component of the magnetic field is one half of this because the x -currents exist for $x < x_c$. That is,

$$B_z(x_c, a < y < b - a, 0) \approx \frac{I}{cy} + \frac{I}{c(b-y)}. \quad (14)$$

The Biot-Savart force on the crosspiece is then approximately,

$$F_x = \int_a^{b-a} \frac{IB_z(x_c, y, 0)}{c} dy \approx \frac{2I^2}{c^2} \ln \frac{b-a}{a} \approx \frac{2I^2}{c^2} \ln \frac{b}{a}, \quad (15)$$

supposing that the wire radius a is small compared to the separation b . Note that even if the current I changes sign as the capacitor discharges, the force is always in the $+x$ direction.

A crosspiece of mass m takes on final x -velocity,

$$v_f = \frac{1}{m} \int F_x dt \approx \frac{2}{mc^2} \ln \frac{b}{a} \int I^2 dt. \quad (16)$$

We can also deduce the force F_x via an energy method, following Art. 686 of [90].

The energy U_C stored in the electric field of the capacitor is,

$$U_C = \frac{Q^2}{2C}. \quad (17)$$

As charge flows off the capacitor (at rate $I = dQ/dt$) the stored energy U_C decreases, being transferred into the energy $U_L = LI^2/2$ stored in the magnetic field, and into the Joule heating (at rate $I^2 R$) and the kinetic energy (at rate $d(mv^2/2)/dt = Fv$) of the crosspiece, assuming that radiation (and the kinetic energy of the electric current) is negligible, where $L(x, t)$ is the self inductance of the circuit when the moving crosspiece is at position $x(t)$, $F(t) = m dv/dt$ is the force on the crosspiece of mass m and velocity $v(t)$ in the $+x$ direction, and we make the approximation that the resistance of the rails is small compared to resistance R of the crosspiece. Conservation of total energy E tells us that,

$$\frac{dE}{dt} = 0 = \frac{dU_C}{dt} + \frac{dU_L}{dt} + I^2 R + F_x v = \frac{QI}{C} + LI \frac{dI}{dt} + \frac{I^2}{2} \frac{dL}{dt} + I^2 R + F_x v, \quad (18)$$

Meanwhile, the circuit equation is, for variable I and variable L ,

$$0 = \frac{Q}{C} + \frac{d(LI)}{dt} + IR. \quad (19)$$

Using this in eq. (18) leads to,

$$\frac{I^2}{2} \frac{dL}{dt} = \frac{I^2}{2} \frac{dL}{dx_c} v = F_x v, \quad (20)$$

such that the force on the crosspiece appears to be given by,

$$F_x = \frac{I^2}{2} \frac{dL}{dx_c}. \quad (21)$$

Here, the current I is not constant, although it is often stated that the form (21) holds only for currents held constant by a battery that does work on the current.

The self inductance of the circuit is related to the magnetic flux Φ through it according to $\Phi = cLI$. We estimate the self inductance supposing the field (13) holds for $0 < x < x_c$,

$$\Phi \approx 2 \int_a^b \frac{2I}{cy} x_c dy = \frac{4x_c I}{c} \ln \frac{b}{a}, \quad (22)$$

and the self inductance L is approximately,²⁴

$$L = \frac{\Phi}{cI} \approx \frac{4x_c}{c^2} \ln \frac{b}{a}. \quad (23)$$

Thus, the force on the crosspiece follows from eq. (21) as,

$$F_x \approx \frac{2I^2}{c^2} \ln \frac{b}{a}, \quad (24)$$

as found previously in eq. (15).

To display an analytic result for the current $I(t)$, we suppose that the crosspiece hardly moves during the current pulse after the capacitor is switched in to the circuit at time $t = 0$, such that the self inductance (23) and the is essentially constant while the crosspiece accelerates to its final velocity. We suppose that the resistance of the crosspiece is large compared to that of the rest of the circuit, such that R is essentially constant as the crosspiece moves. We also suppose that the time constant L/R for the current to rise in the inductance L is small compared to the time constant RC for the capacitance C to discharge, when in series with resistance R . Then, the circuit equation (19) takes the more usual form,

$$0 = \frac{Q}{C} + L \frac{dI}{dt} + IR, \quad (25)$$

for which a trial solution $I = e^{\alpha t}$, and the condition that $I(0) = 0$, lead to,

$$I(t > 0) \approx A (e^{-t/RC} - e^{-Rt/L}). \quad (26)$$

The determine the constant A , we integrate the current (26), which must equal the initial charge Q on the capacitor,

$$Q = \int_0^\infty I dt \approx A \left(RC - \frac{L}{R} \right) \approx ARC, \quad A \approx \frac{Q}{RC}, \quad (27)$$

and the current waveform is approximately,

$$I(t > 0) \approx \frac{Q}{RC} (e^{-t/RC} - e^{-Rt/L}), \quad I(t \gtrsim L/R) \approx \frac{Q}{RC} e^{-t/RC}. \quad (28)$$

The final velocity of the crosspiece follows from eqs. (16)-(17) as,

$$v_f \approx \frac{Q^2}{mc^2 RC} \ln \frac{b}{a}, \quad \frac{v_f}{c} = \frac{6}{R[\Omega]} \frac{U_C}{mc^2} \ln \frac{b}{a}, \quad (29)$$

noting that a resistance R in Gaussian units is the resistance in Ohms divided by $3c$.

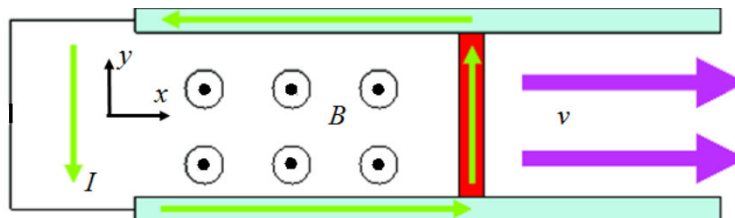
For example, with $U_C = 10^4$ J, a 1-cm³ copper cube of mass 9 g and resistance $R \approx 2 \times 10^{-6} \Omega$, would be accelerated to $v_f \approx 2.5 \times 10^{-4} c$ by the railgun, supposing the rails are 1 cm² copper bars such that $b/a = 2$ and that friction can be ignored. This is close to the escape velocity (11 km/s $\approx 3 \times 10^{-4} c$) of the Earth.

²⁴See Art. 685 of [90] for a slightly better approximation.

2.4.2 Initially Superconducting Railgun

In the variant with an initially superconducting railgun, where the crosspiece goes normal, and is set free, at time $t = 0$, the Lorentz force on the crosspiece is again given by eq. (15), and the self inductance is again given by eq. (23). For $t > 0$, the circuit equation is $L dI/dt + IR = 0$ in the approximation that the crosspiece does not move significantly as the current decays to zero, such that the current falls off exponentially,

$$I(t) = I_0 e^{-Rt/L}. \quad (30)$$



The final velocity of the crosspiece follows from eqs. (16), (23) and (30) as,

$$v_f \approx \frac{2}{mc^2 R} \ln \frac{b}{a} \frac{U_0}{R} \approx \frac{I_0^2 L}{mc^2 R} \ln \frac{b}{a} \approx \frac{4I_0^2 x_c}{mc^4 R} \ln^2 \frac{b}{a}. \quad (31)$$

where $U_0 = I_0 L^2/2$ is the initial stored magnetic-field energy.

The final velocity (31) is small, but nonzero, and the nonzero final kinetic energy of the crosspiece came from the initial energy stored in the magnetic field of the circuit.²⁵

However, the statement “magnetic forces do no work” seems to imply that the crosspiece could not move in this case.

2.4.3 Three Commentaries

1. The Abstract of [161] claims:²⁶

Magnetic forces can never do work. The force the conductor exerts on the electrons is shown to do the work.

And in the third paragraph of the main text of [161] one reads:

Many undergraduate physics texts, however, refer to $\mathbf{v} \times \mathbf{B}$ as a magnetic force. Such reference implies that a magnetic force does work on a charge traversing the moving rod. This of course does not happen. (If it did happen the magnet would be supply energy to the circuit. The magnet, since it would then be losing energy, would become colder. This is not observed.) As will be shown, the force $\mathbf{v} \times \mathbf{B}$ is the sum of two forces, the force the rod exerts on the charge and the magnetic force.

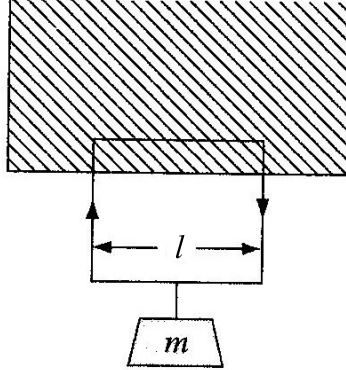
2. At the end of sec. II of [165] the statement is made:

²⁵If the crosspiece did not move after going “normal”, all of the initial magnetic energy, U_0 , would be dissipated as Joule heating, as the time integral of eq. (30) is U_0/R . If the crosspiece moves, the self inductance L increases with time, the current drops more quickly than the estimate of eq. (30), and the final velocity is somewhat less than the estimate of eq. (31).

²⁶Ref. [285] is similar to [161], but claims the $I d\mathbf{l} \times \mathbf{B}$ force is (electro)static.

A magnetic field does not give a current carrying wire any energy by doing work on it although it indirectly causes the net force on the wire.

3. A popular textbook on electrodynamics [271] prominently displays the statement “magnetic forces do no work” in sec. 5.1.2, with the implication that in cases where it seems that a magnetic force does work, actually some other force did the work. In ex. 5.2 of [271], a railgun variant in which a current loop is pulled into a region of external magnetic field,



it is stated:

Was work done in this process? Absolutely! Who did it? The battery!

However, the example need not involve a battery, as for the cases of sec. 2.4.1 and 2.4.2 above.²⁷

All three commentators could be interpreted as implying that the crosspiece would not move in the example of sec. 2.4.2 since (they appear to claim) a magnetic field cannot transfer energy to (do work on) an electromechanical system.

It is, of course, hard to know what these authors actually mean by their comments, but in view of their provocative nature, we continue with some comments of our own.

2.4.4 Transverse Forces on the Current and the Lattice of the Crosspiece

We now examine the forces on the crosspiece in more detail. In so doing, we depart from the analyses of Ampère and of Maxwell by supposing that the electric current is associated with moving charges inside the conductors, which also support charges at rest relative to the conductors, and we consider the Lorentz force (3) on these charges.

The bulk matter²⁸ of the crosspiece (and of the rest of the circuit) can be partitioned into the moving electrons of the electric current I and the “lattice” ions which are at rest in the rest frame of the crosspiece. The moving electrons (in the crosspiece) have charge density $\rho^- < 0$ and average drift velocity $\mathbf{v}_d = -v_d \hat{\mathbf{y}}$ ($0 < v_d < 1$ cm/s), and the lattice ions have charge density $\rho^+ = -\rho^-$.²⁹

²⁷In any case, the work done by the battery is on the electric charges that move through it, and not on distant objects.

²⁸There are also electric charges on the surface of the crosspiece, as mentioned below.

²⁹There is actually a tiny difference, of order $1/c^2$, between ρ^- and ρ^+ [269] that we ignore here.

If the only force on the current I were the Lorentz force F_x identified in eq. (15), then the current would be forced out of the crosspiece, which does not happen. Rather, there must exist a force $F_{H,x} \approx -F_x$ on the current exerted by the crosspiece, which keeps the current inside the latter. In turn, the current exerts a force $F'_{H,x} = -F_{H,x} \approx F_x$ on the lattice ions of the crosspiece, and it is the force $F'_{H,x}$ that accelerates the lattice to the final velocity v_f .

2.4.5 The Hall Field

The force $F_{H,x}$ is said in [139, 157, 161, 165] to be due to the so-called Hall electric field \mathbf{E}_H , first noted Hall in 1879 [92], which view is a reasonable approximation for currents associated with “free” electrons [157, 167]. In the limit of negligible electron mass (or for steady currents), the transverse force on the moving electrons (of electric charge e) must be zero, $0 \approx e(\mathbf{E}_H + \mathbf{v}_e/c \times \mathbf{B})$,

$$\mathbf{E}_H \approx -\frac{\mathbf{v}_e}{c} \times \mathbf{B} \approx -\frac{v\hat{\mathbf{x}} - v_d\hat{\mathbf{y}}}{c} \times B_z\hat{\mathbf{z}} = \frac{v_d B_z}{c}\hat{\mathbf{x}} + \frac{v B_z}{c}\hat{\mathbf{y}} \equiv E_{H,x}\hat{\mathbf{x}} + E_{H,y}\hat{\mathbf{y}} \quad (32)$$

where we approximate that magnetic field on the electrons by eq. (14), ignoring the field due to the current in the crosspiece. The Hall field (32) is maintained by an appropriate charge density on the surface of the crosspiece.

The component $E_{H,x}$ implies that a potential difference $2av_d B_z/c$ exists in the x -direction between the two sides of the crosspiece, whose sign and magnitude permit one to infer the (small) value of v_d and to confirm that the sign of the moving charges is negative. Similarly, the component $E_{H,y}$ implies that a potential difference,

$$V_{\text{motional}} = \frac{v}{c} \int_a^{b-a} B_z dy \quad (33)$$

exists in the y -direction between the two ends of the crosspiece. This effect was noted by Faraday in secs. 3078 and 3087 of [73], and is often called the **motional EMF**.³⁰

The forces $F_{H,x}$ and $F'_{H,x}$ can now be related as,

$$F_{H,x} = \int_c \rho^- E_{H,x} d\text{Vol} \approx -F_x, \quad F'_{H,x} = \int_c \rho^+ E_{H,x} d\text{Vol} = -F_{H,x} \approx F_x. \quad (34)$$

The magnetic force F_x of eq. (15) does work on the moving crosspiece at rate,³¹

$$\frac{dW_x}{dt} = F_x v, \quad (35)$$

while the Hall field $E_{H,x}$ does work on the moving electrons at rate,

$$\frac{dW_{H,x}}{dt} = F_{H,x} v \approx -F_x v = -\frac{dW_x}{dt}, \quad (36)$$

³⁰A recent article [280] on Ampère’s “hairpin” experiment includes measurement of the motional EMF.

³¹Work is, of course, a scalar quantity. the subscript x indicates that the work is done by a force in the x -direction.

and the Hall field $E_{H,x}$ does work on the moving lattice at rate,

$$\frac{dW'_{H,x}}{dt} = F'_{H,x}v \approx -F_{H,x}v = -\frac{dW_{H,x}}{dt} \approx \frac{dW_x}{dt}. \quad (37)$$

The total work done by the Hall field $E_{H,x}$ is zero, and the sum of the work done by the magnetic force F_x and by the Hall field $E_{H,x}$ is just the work done by the magnetic force F_x .

Since the total work done by the Hall field $E_{H,x}$ is zero, it would be consistent with the statement “magnetic forces do no work”, to say that “the Hall force does no work”. However, the attitude of [161, 165] seems to be that the Hall field, and not the magnetic force F_x , does the work of accelerating the crosspiece, because it is the Hall field and not the magnetic force F_x that acts directly (in the x -direction) on the lattice ions. The latter is technically correct, but does not imply that the magnetic force F_x does no work.³²

2.4.6 Axial Forces in the Crosspiece

The Lorentz force due to the magnetic field B_z on the conduction electrons in the crosspiece has an axial component,

$$F_y = \int_c \frac{\rho^- v \hat{\mathbf{x}} \times B_z \hat{\mathbf{z}}}{c} d\text{Vol} = \int_c \frac{-\rho^- v B_z}{c} d\text{Vol} = \frac{v}{v_d} \int_a^{b-a} \frac{I B_z}{c} dy = \frac{v}{v_d} F_x, \quad (38)$$

noting that the current is related by $I = \rho^-(-v_d)A > 0$, where A is the effective area of the conduction current density perpendicular to the y -axis.³³ The magnetic force F_y is in the opposite direction to the y -velocity of the conduction electrons, so it does work at the rate,

$$\frac{dW_y}{dt} = F_y(-v_d) = -F_x v = -dW_x/dt. \quad (39)$$

The total rate of work done by magnetic force on the current in the crosspiece is zero,

$$\frac{dW_{\text{total}}}{dt} = \frac{dW_x}{dt} + \frac{dW_y}{dt} = F_x v - F_x v = 0. \quad (40)$$

This is rather different that the statement “magnetic forces do no work”, which seems to imply that $dW_x/dt = 0 = dW_y/dt$.

Now, dW_x/dt is a large quantity, being equal to the rate of change of kinetic energy of the crosspiece, whose motion is in the direction of the magnetic force F_x . So, dW_y/dt is also a large (negative) quantity, but it does not lead to any noticeable change in energy of the system. Rather, it serves to cancel the large positive rate of work done by the Hall field $E_{H,y}$ on the conduction electrons,

$$F_{H,y} = \int_c \rho^- E_{H,y} d\text{Vol} = \int_a^{b-a} \frac{I v B_z}{(-v_d)c} dy = -\frac{v}{v_d} F_x = -F_y, \quad (41)$$

³²In the opinion of this author, [161, 165] (and also [271]) made an unfortunate play on words in an effort to support the claim that “magnetic fields never do any work”. The discussion there is akin to concluding from the fact that $1 - 1 = 0$ that $1 = 0 = -1$.

³³The effective area A is less than the geometric area πa^2 due to the skin effect for the transient current.

$$\frac{dW_{H,y}}{dt} = F_{H,y}(-v_d) = F_x v = -\frac{dW_y}{dt}. \quad (42)$$

Because the rate of work done by the axial component of the Hall field, $E_{H,y}$, equals the rate of change of kinetic energy of the crosspiece, [161, 165] seems to claim that this field is the “cause” or the “source of energy” for the increasing kinetic energy.³⁴ This is poor logic, arguing that if $A = B$, then A “causes” or is the “source of” B .³⁵ The Hall field \mathbf{E}_H does not exist in the initial state of the railgun of sec. 2.4.1, so cannot be the “source” of the final kinetic energy of the crosspiece.

There are several more axial forces on the current, and on the lattice, of the crosspiece to be discussed.

The Hall field $E_{H,y}$ exerts a force on the lattice ions,

$$F'_{H,y} = \int_c \rho^+ E_{H,y} d\text{Vol} = -F_{H,y} = F_y. \quad (43)$$

This force does no work, as the lattice ions do not move in the y -direction; $dW'_{H,y}/dt = 0$.

The crosspiece has resistance R which implies that the lattice exerts a retarding force $F_{R,y}$ on the conduction electrons. Of course, these electrons would soon come to rest if there were not a force $F_{\sigma,y}$ driving the current in the y -direction inside the crosspiece. This latter force is due to the internal axial electric field \mathbf{E}_σ related by Ohm’s law,³⁶

$$\mathbf{J} = \sigma \mathbf{E}_\sigma, \quad E_{\sigma,y} = \frac{J_y}{\sigma} = \frac{I}{\sigma A}, \quad V_\sigma = E_{\sigma,y} b = I \frac{b}{\sigma A} = IR, \quad (44)$$

where σ is the electrical conductivity of the crosspiece, related to its resistance by $R = b/\sigma A$. The axial force $F_{\sigma,y}$ that drives the conduction electrons is,

$$F_{\sigma,y} = \int_c \rho^- E_{\sigma,y} d\text{Vol} = \frac{I b E_{\sigma,y}}{(-v_d)} = \frac{I^2 R}{(-v_d)}, \quad (45)$$

so this force does work on the current at rate,

$$\frac{dW_{\sigma,y}}{dt} = F_{\sigma,y}(-v_d) = I^2 R, \quad (46)$$

which equals the rate of Joule heating of the resistive crosspiece.

The retarding force $F_{R,y}$ on the conduction electrons is, of course, related by,

$$F_{R,y} = -F_{\sigma,y}. \quad (47)$$

This force is due to collisions, and does not do work, although it is associated with reducing the kinetic energy of the conduction electrons; $dW_{R,y}/dt = 0$.

³⁴It is claimed in [280] that the Hall field (there called the motional EMF) serves as an “intermediary” and “delivers lift to the wire”. This also is misleading, as it is the Poynting vector, which depends on the total \mathbf{E} and total \mathbf{B} fields, that delivers the energy from the battery to the moving wire in that example.

³⁵The statement “magnetic forces do no work” seems to inspire faulty logic, which is perhaps the author’s main objection to it.

³⁶The original version of Ohm’s law [52] was closer in form to the first two lines of eq. (44) than to the third.

The collision of the conduction electrons with the lattice result in a force $F'_{R,y}$ on the latter,

$$F'_{R,y} = -F_{R,y} = F_{\sigma,y}, \quad (48)$$

and this collision force also does no work, while increasing the thermal energy of the lattice; $dW'_{R,y}/dt = 0$.

The axial electric field \mathbf{E}_σ exerts force $F'_{\sigma,y}$ on the lattice ions,

$$F'_{\sigma,y} = \int_c \rho^+ E_{\sigma,y} d\text{Vol} = -F_{\sigma,y}. \quad (49)$$

This force does no work, as the lattice ions do not move in the y -direction; $dW'_{\sigma,y}/dt = 0$.

Furthermore, there is a Lorentz force on the lattice ions associated with their velocity $v \hat{\mathbf{x}}$ in the magnetic field B_z ,

$$F'_y = \int_c \frac{\rho^+ v \hat{\mathbf{x}} \times B_z \hat{\mathbf{z}}}{c} d\text{Vol} = - \int_c \frac{\rho^+ v B_z}{c} d\text{Vol} = \int_c \frac{\rho^- v B_z}{c} d\text{Vol} = -F_y. \quad (50)$$

This force does no work, as the lattice ions do not move in the y -direction; $dW'_y/dt = 0$.

There are also forces associated with the deflection of the current by 90° at the ends of the crosspiece, but we neglect these here.

2.4.7 The Total Force

We have now identified 11 force components on the crosspiece:

- | | |
|---|--------------------------------|
| 1. F_x = magnetic force on the current. | $\dot{W}_x = dW_x/dt = F_x v.$ |
| 2. $F_{H,x} = -F_x$ = electric force on the current due to the Hall field. | $\dot{W}_{H,x} = -F_x v.$ |
| 3. $F'_{H,x} = F_x$ = electric force on the lattice due to the Hall field. | $\dot{W}'_{H,x} = F_x v.$ |
| 4. $F_y = F_x v/v_d$ = magnetic force on the current. | $\dot{W}_y = -F_x v.$ |
| 5. $F'_y = -F_y$ = magnetic force on the lattice. | $\dot{W}'_y = 0.$ |
| 6. $F_{H,y} = -F_y$ = electric force on the current due to the Hall field. | $\dot{W}_{H,y} = F_x v.$ |
| 7. $F'_{H,y} = F_y$ = electric force on the lattice due to the Hall field. | $\dot{W}'_{H,y} = 0.$ |
| 8. $F_{\sigma,y} = -I^2 R/v_d$ = electric force on the current due to the field $\mathbf{E}_\sigma = \mathbf{J}/\sigma$. | $\dot{W}_{\sigma,y} = I^2 R.$ |
| 9. $F'_{\sigma,y} = -F_{\sigma,y}$ = electric force on the lattice due to the field \mathbf{E}_σ . | $\dot{W}'_{\sigma,y} = 0.$ |
| 10. $F_{R,y} = -F_{\sigma,y}$ = force on the current due to collisions with the lattice. | $\dot{W}_{R,y} = 0.$ |
| 11. $F'_{R,y} = -F_{R,y}$ = force on the lattice due to collisions with the current. | $\dot{W}'_{R,y} = 0.$ |

The total force in the x -direction is,

$$F_{x,\text{total}} = F_x + F_{H,x} + F'_{H,x} = F_x - F_x + F_x = F_x \approx \frac{2I^2}{c^2} \ln \frac{b}{a}, \quad (51)$$

and the total rate of work done by the forces in the x -direction is,

$$\frac{dW_{x,\text{total}}}{dt} = \frac{dW_x}{dt} + \frac{dW_{H,x}}{dt} + \frac{dW'_{H,x}}{dt} = F_x v - F_x v + F_x v = F_x v. \quad (52)$$

The total force in the y -direction is,

$$\begin{aligned} F_{y,\text{total}} &= F_y + F'_y + F_{H,y} + F'_{H,y} + F_{R,y} + F'_{R,y} + F_{\sigma,y} + F'_{\sigma,y} \\ &= F_y - F_y - F_y + F_y + F_{R,y} - F_{R,y} - F_{R,y} + F_{R,y} = 0, \end{aligned} \quad (53)$$

so the crosspiece does not move in the y -direction even if it were not between side rails, but rather lying on top of them. The total rate of work done by the forces in the y -direction is,

$$\frac{dW_{y,\text{total}}}{dt} = \frac{dW_y}{dt} + \frac{dW_{H,y}}{dt} + \frac{dW_{\sigma,y}}{dt} = -F_x v + F_x v + I^2 R = I^2 R. \quad (54)$$

The total force on the system is in the $+x$ direction,

$$F_{\text{total}} = F_x \hat{\mathbf{x}} = F'_{H,x} \hat{\mathbf{x}}. \quad (55)$$

and is equal to the x -component of the magnetic force on the conduction electrons. The total work done on the system is

$$\frac{dW_{\text{total}}}{dt} = \frac{dW_{x,\text{total}}}{dt} + \frac{dW_{y,\text{total}}}{dt} = F_x v + I^2 R. \quad (56)$$

The rate of increase of kinetic energy equals the rate of work done by the x -component of the magnetic force on the conduction electrons, and the Joule heating equals the rate of work done by the (partial) axial electric field \mathbf{E}_σ on these electrons.³⁷

Are There 11 Forces, or Is There Just One, Total Force?

From the macroscopic point of view (that of Ampère and Maxwell, for example), only the magnetic force $F_x \hat{\mathbf{x}}$ exists. The other 10 forces arise only in a microscopic view in which the crosspiece is partitioned into the conduction current and the lattice. And, of these 10 forces, 8 are internal to the crosspiece and, of course, sum to zero. Hence, it seems to this author that there is merit in emphasizing the macroscopic view, in which “the” force on the crosspiece is the magnetic force $F_x \hat{\mathbf{x}}$, and to say that this is the force that does the work which increases the (macroscopic) kinetic energy of the crosspiece.

However, opinions differ on this issue, and as a consequence, attitudes vary towards the phrase “magnetic forces do no work”.

³⁷An accounting like the above appeared in [270], but with only 5 force components rather than 11.

Is the Total Force Magnetic or Electric?

The total force equals $F_x \hat{\mathbf{x}}$ which is a (macroscopic) magnetic force, but this also equals $F'_{H,x} \hat{\mathbf{x}}$ which is a (somewhat subtle, microscopic) electric force due to the Hall field.

The total work done by the three magnetic force components F_x , F_y and F'_y is zero, so some people say that “magnetic forces do no work”, even though both forces F_x and F_y do (equal and opposite) work.

The total work done by the four Hall force components $F_{H,x}$, $F_{H,y}$, $F'_{H,x}$ and $F'_{H,y}$ is $F_x v = F'_{H,x} v = F_{H,y} v$, so some people say that the Hall field does the work in this problem, but they disagree as to which component of the Hall force does the work: [161] claims that $F'_{H,x}$ does the work, while [271] claims that $F_{H,y}$ does it (with the implication that the total force $F_x \hat{\mathbf{x}}$ on the crosspiece did not do the work on it).

2.4.8 Separate Accounting for the Current and the Lattice of the Crosspiece

The forces on the current in the crosspiece, discussed in secs. 2.4.4-2.4.6, sum to,

$$\begin{aligned} \mathbf{F}_{\text{current}} &= (F_x + F_{H,x}) \hat{\mathbf{x}} + (F_y + F_{H,y} + F_{R,y} + F_{\sigma,y}) \hat{\mathbf{y}} \\ &\approx (F_x - F_x) \hat{\mathbf{x}} + (F_y - F_y - F_{\sigma,y} + F_{\sigma,y}) \hat{\mathbf{y}} = 0, \end{aligned} \quad (57)$$

in the approximation that the current is massless, such that any force on it must be zero. The rate of work done by these forces is,

$$\begin{aligned} \frac{dW_{\text{current,total}}}{dt} &= \frac{dW_x}{dt} + \frac{dW_{H,x}}{dt} + \frac{dW_y}{dt} + \frac{dW_{H,y}}{dt} + \frac{dW_{\sigma,y}}{dt} \\ &\approx F_x v - F_x v - F_x v + F_x v + I^2 R = I^2 R. \end{aligned} \quad (58)$$

The forces on the lattice ions sum to,

$$\mathbf{F}_{\text{lattice}} = F'_{H,x} \hat{\mathbf{x}} + (F'_y + F'_{H,y} + F'_{R,y} + F'_{\sigma,y}) \hat{\mathbf{y}} \approx F_x \hat{\mathbf{x}} + (F_y - F_y + F_{\sigma,y} - F'_{\sigma,y}) \hat{\mathbf{y}} = F_x \hat{\mathbf{x}}, \quad (59)$$

and the rate of work done by these forces is,

$$\frac{dW_{\text{lattice,total}}}{dt} = \frac{dW'_{H,x}}{dt} \approx F_x v. \quad (60)$$

The total rate of work done on the crosspiece is again given by eq. (56).

2.4.9 Ampère Tension in the Crosspiece?

A key result of sec. 2.2 was that the magnetic force on a “current element” is perpendicular to that element. In contrast, the analysis of sec. 2.4.6 began with the claim that the magnetic force on the crosspiece has an axial component F_y , eq. (38). However, the axial magnetic force F_y is a force on the conduction current, which is distinct from the lattice of the crosspiece in a microscopic view.

There is also a magnetic force $F'_y = -F_y$ on the lattice. While the total axial magnetic force on the crosspiece is zero, the nonzero, axial, magnetic force F'_y on the lattice can be considered as a kind of “Ampère tension” in the microscopic view.

Of course, the axial force F'_y is counteracted by the Hall force $F'_{H,y} = F_y$ on the lattice, such that the nominal Ampère tension F'_y does not lead to any axial stress on the lattice.

Recall that in the macroscopic view of Ampère (and Maxwell), a “current element” is electrically neutral and is not partitioned into charges moving and fixed (relative to the element), such that F_y and $F'_y = -F_y$ do not exist in this view, and the only magnetic force on the crosspiece “current element” is perpendicular to it, *i.e.*, F_x . Indeed, in the macroscopic view, F_x is the only force on the crosspiece, and it does the work on the crosspiece.

2.4.10 The Lorentz Force

In secs. 2.4.4-2.4.7 we have considered 9 Lorentz-force components on the crosspiece, the magnetic forces F_x , F_y (on the current), F'_y (on the lattice), the electric forces $F_{H,x}$, $F_{H,y}$, $F_{\sigma,y}$ (on the current), and $F'_{H,x}$, $F'_{H,y}$, $F'_{\sigma,y}$ (on the lattice), as well as the two collision-related forces $F_{R,y}$ (on the current) and $F'_{R,y}$ (on the lattice). The total Lorentz force is,

$$\begin{aligned}\mathbf{F}_{\text{Lorentz}} &= (F_x + F_{H,x} + F'_{H,x})\hat{\mathbf{x}} + (F_y + F'_y + F_{H,y} + F'_{H,y} + F_{\sigma,y} + F'_{\sigma,y})\hat{\mathbf{y}} \\ &\approx F_x\hat{\mathbf{x}} = F'_{H,x}\hat{\mathbf{x}}.\end{aligned}\tag{61}$$

As noted above, opinions differ as to whether the total Lorentz force on the crosspiece is electric or magnetic. This author prefers the view that the total Lorentz force (which is also the total force on the crosspiece) is a magnetic force.

2.4.11 What’s Going On Here?

Macroscopic *vs.* Microscopic Views

The controversy over the statement “magnetic forces do no work” has its origin in the difference between macro- and microscopic views of electrical currents. In the macroscopic view of Ampère and Maxwell, currents in electrical circuits are electrically neutral, and the magnetic force on a current element is perpendicular to that element. If that element moves in response to this magnetic force, the magnetic force does work [286]. In the microscopic view that electrical current is due to moving charges, against a “lattice” of positive charge that does not move relative to a current element, one considers separately the magnetic forces on the positive and negative charges, and the **total** work done by the magnetic forces on each charge is zero.

The microscopic view is fundamentally appealing, but it is not evidently true for ordinary electrical circuits, and took ≈ 60 years, from the time of Ampère to that of Lorentz to become accepted (and even today seems “impractical” to those particularly concerned with “circuit analysis”.³⁸). So, while the statement “magnetic forces do no work” has its place, it would be better if the statement were more like “In a microscopic view, the total magnetic force on a moving electric charge does no work, but the macroscopic components of the magnetic force can do work, and magnetic fields can transfer energy to/from moving charges.”

Work Is Not Energy

³⁸In the example of a railgun, with 11 forces on the crosspiece, 10 of which cancel internally, the latter forces are typically poorly represented in the technical literature reviewed in this note.

The textbook [271] (and also [280]) is insufficiently clear as to the distinction between work and energy. Work is a “local” quantity associated with a force F at some point/volume whose point/volume of application moves through distance dx , $dW = F dx$, while energy is a more “global” quantity that can exist in the electromagnetic field throughout all space, as well as in other forms.

The energy needed to perform the work at point A can be due to a source at point B, with the energy being transmitted from B to A via the electromagnetic field.

It is a colloquialism to say that the work done at point A was done by the source at point B, rather than by the force F_A at point A. A more proper technical description is that the energy source at point B provided the energy for the work done by force F_A at point A.

The Poynting Vector Describes the Flow of Energy in the Electromagnetic Field

In addition to being enmeshed in the ambiguity as to the character of the total force, and the issue of which force does “the” work, the discussion of all three commentators [161, 165, 271] (and also that in [280]) gives the impression that railguns can’t work unless they contain a “battery”, whereas the preferred energy source for contemporary railguns is a capacitor bank [272].

What’s missing is an awareness that energy transfer in electromagnetic systems is mediated by the Poynting vector, eq. (1), and not by forces inside wires.³⁹ The source of the Poynting vector can be nonelectromagnetic energy stored in a “battery”, but it can also be energy stored in the electromagnetic field.⁴⁰ In any case, energy can be transferred by an electromagnetic field only when both \mathbf{E} and \mathbf{B} fields are present. Hence, when seeking detailed understanding of the flow of energy in an electromechanical system, one should not focus only on the \mathbf{E} field, or only on the \mathbf{B} field, but rather one must consider the full electromagnetic field.

The statement “magnetic fields do no work” serves as an indirect reminder that one should consider the electric field as well, but in this author’s view, it is more pertinent to note that the Poynting vector (1) transmits the energy that “does the work”.

³⁹As remarked by Sommerfeld on p. 173 of [140]:

Conductors are nonconductors of energy. Electromagnetic energy is transported without loss only in nonconductors; in conductors it is destroyed, or rather transformed. The notation “conductor” and “nonconductor” refers only to the behavior with respect to charge; it is misleading if applied to behavior with respect to energy.

⁴⁰See, for example, sec. 2.5 of [266].

A Appendix: Historical Survey of (Electro)Magnetism

This Appendix is superceded by the Appendix to [284].

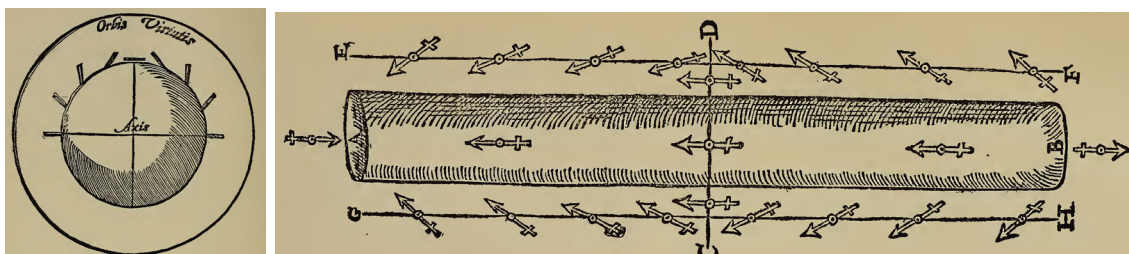
A.1 Peregrinus

In 1269, Peregrinus [1] concluded from experiment that a (permanent) magnet has two “poles”, and that like poles of different magnets repel, while unlike poles attract. Peregrinus’ methods were later notably extended by Gilbert.

A.2 Gilbert

In 1600, Gilbert published a treatise [2], which includes qualitative notions of magnetic energy and lines of force.⁴¹ These seem to have been inspired by experiments (suggested by Peregrinus) in which magnetized needles (or steel filings, p. 162) were used to probe the space around larger magnets, particularly spherical magnets called *terrellas*. The observed directions of the needle (or filings) suggested the existence of lines of force throughout space, and the ability of the magnet to deflect the needle into alignment with them suggested (to Gilbert) that some kind of magnetic energy exists outside the magnet itself.

The figures below are from p. 122 and 247 of [2].

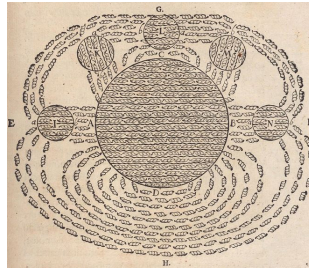


A.3 Descartes

In 1644, Descartes published a qualitative treatise on physics science [3]. Among its notable features is perhaps the earliest conception of momentum (mass times velocity, p. 59 ff),⁴² and that light could be due to static pressure in a kind of elastic medium that fills all space, later called the *æther* (pp. 94-104, part III). And, on p. 271, part IV, he presented a figure based on use of iron filings near a magnet which illustrates Gilbert’s lines of magnetic force.

⁴¹Then, as now, magnetism seems to have inspired claims of questionable merit, which led Gilbert to pronounce on p. 166, “May the gods damn all such sham, pilfered, distorted works, which do but muddle the minds of students!”

⁴²The next use of momentum may be by Wallis and Wren in 1668 [4], and by Huyghens in 1669 [5].



A.4 Hooke

In 1667, Hooke (p. 55 ff of [6]) argued that light is a wave phenomenon in an elastic medium.

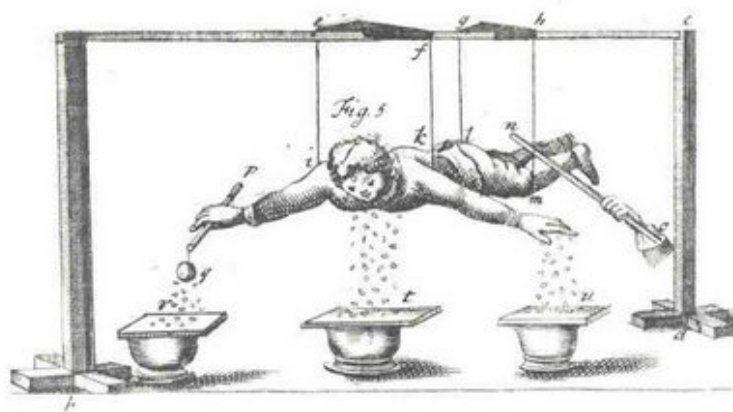
A.5 Newton

In 1686, Newton published his *Principia*, in which comments on p. 246 of [7] are usually interpreted as support for a particle (rather than wave) theory of light, although Newton calls this only an “analogy”. He did not pursue the implication that light would carry energy and momentum (concepts that were not yet well developed).

Newton also deduced that the attractive force of gravity varies with distance between the centers of two masses as m_1m_2/r^2 (see the Scholium on p. 392 of [7]).

A.6 Gray

The phenomenon of conduction of electricity may have been first reported by Gray in 1731 [8], who described experiments on the “conveyance” and “communication” of electricity (including evidence that people conduct electricity).⁴³



In 1739, Desaguliers [10] (a coworker of Gray) coined the term “conductor”, as well as the term “insulator”.

⁴³For a historical survey of research into electricity in this era, see, for example, [166].

A.7 Michell

In 1750, Michell published a treatise on the manufacture of magnets [11], including on p. 19 the statement that the repulsive force between like poles of two magnets fall off as $1/r^2$.⁴⁴

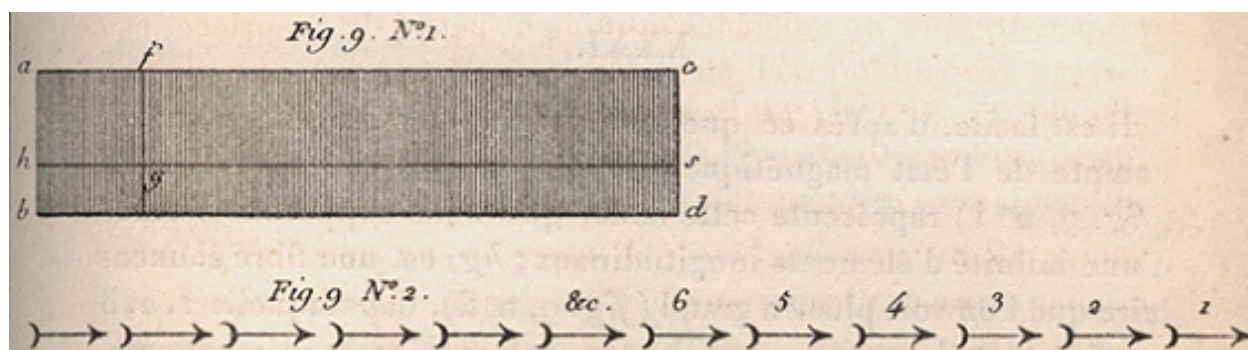
A.8 Priestley

In 1766, Priestley [13] deduced that the static force between electric charges varies as $1/r^2$, similar to the force of gravity except that like charges repel rather than attract.

A.9 Coulomb

In 1785, Coulomb confirmed (and made widely known) that the static force between pairs of electric charges q_1 and q_2 varies as q_1q_2/r^2 [14], and that the force between idealized magnetic poles p_1 and p_2 at the ends of long, thin magnets varies as p_1p_2/r^2 [15]. The electric and magnetic forces were considered to be unrelated, except that they obeyed the same functional form.

Coulomb also noted that magnetic poles appear not to be isolatable, conjecturing (p. 306 of [16]) that the fundamental constituent of magnetism, a *molécule de fluide magnétique*, is a dipole, such that effective poles appear at the ends of a long, thin magnet.



Coulomb's argument is much superior to that by Gilbert, p. 247 of [2].

A.10 Poisson

In 1812, Poisson [17] extended the use (by Lagrange and Laplace) of a potential $V = -Gm/r$ (= energy per unit mass) for the gravitational force between of a mass m and a unit test mass to the case of static electrical forces, and in 1821-23 for static magnetic forces [36, 42].⁴⁵

⁴⁴Michell is also credited with being the first to discuss what are now called black holes [12].

⁴⁵Following the precedent from gravity, Poisson did not appear to ask where the configuration energy, such as q_1q_2/r and p_1p_2/r , resided, nor did he consider the quantity $-\nabla V$ to be a force field in the space outside the relevant charges or poles.

A.11 Ørsted

In 1820, Ørsted [18, 19] published decisive evidence that electric currents exert forces on permanent magnets, indicating the electricity and magnetism are related.⁴⁶

Ørsted’s term “electric conflict”, used in his remarks on p. 276 of [19], is a precursor of the later concept of the magnetic field.

It is sufficiently evident from the preceding facts that the electric conflict is not confined to the conductor, but dispersed pretty widely in the circumjacent space. From the preceding facts we may likewise infer that this conflict performs circles.

Ørsted also argued from Newton’s 3rd law of action and reaction that a magnet should exert a force on an electric current, and published evidence for this in [20].

A.12 Biot and Savart

Among the many rapid responses in 1820 to Ørsted’s discovery was an experiment by Biot and Savart [21, 24] on the force due to an electric current I in a wire on one pole, p , of a long, thin magnet. The interpretation given of the result was somewhat incorrect, which was remedied by Biot in 1821 and 1824 [28, 46] with a form that can be written in vector notation (and in SI units) as,

$$\mathbf{F}_{\text{on } p} = \frac{\mu_0 p}{4\pi} \oint \frac{I d\mathbf{l} \times \hat{\mathbf{r}}}{r^2}, \quad (62)$$

where \mathbf{r} is the distance from a current element $I d\mathbf{l}$ to the magnetic pole. There was no immediate interpretation of eq. (62) in terms of a magnetic field, $\mathbf{B} = \mathbf{F}/p$, due to the electric current I ,⁴⁷

$$\mathbf{B} = \frac{\mu_0}{4\pi} \oint \frac{I d\mathbf{l} \times \hat{\mathbf{r}}}{r^2}, \quad (63)$$

which expression is now commonly called the Biot-Savart law.⁴⁸

Biot and Savart did not discuss the force on an electric current, but if we suppose the force on pole p due to the electric current I is equal and opposite to the force of the pole on the electric current, and note that the magnetic field of the pole at the current element $I d\mathbf{l}$ is $\mathbf{B} = -\mu_0 p \hat{\mathbf{r}}/4\pi r^2$ for $\hat{\mathbf{r}}$ as in eq. (62), then,

$$\mathbf{F}_{\text{on } I} = \oint I d\mathbf{l} \times \mathbf{B}, \quad (64)$$

⁴⁶Reports have existed since at least the 1600’s that lightning can affect ship’s compasses (see, for example, p. 179 of [86]), and an account of magnetization of iron knives by lightning was published in Phil. Trans. in 1735 [9]. In 1797, von Humboldt conjectured that certain patterns of terrestrial magnetism were due to lightning strikes (see p. 13 of [265], a historical review of magnetism). A somewhat indecisive experiment involving a voltaic pile and a compass was performed by Romagnosi in 1802 [257].

Historical commentaries on Ørsted’s work include [141, 146, 283].

⁴⁷Although the concept of the magnetic field is latent in discussions of magnetic force by Michell, Coulomb, Poisson and Ørsted (and many other in the years 1820-45), the first use of the term “magnetic field” seems to be due to Faraday, Art. 2247 of [66] (1845).

⁴⁸An early example is on p. 220 of [110].

which is now often called the Biot-Savart force law.⁴⁹ Finally, we can consider the magnetic field \mathbf{B} in eq. (64) to be that due to a second loop with steady current, as in eq. (63) with $I \rightarrow I_2$, such that the combination of these two equations gives the force of one circuit on another, when they both carry steady currents. See also eq. (71) below.

A.13 Ampère

Between 1820 and 1825 Ampère made extensive studies [22, 23, 25, 26, 27, 29, 30, 31, 32, 33, 37, 38, 44, 45, 47, 48, 51] of the magnetic interactions of electrical currents.⁵⁰ Already in 1820 Ampère came to the vision that all magnetic effects are due to electrical currents.^{51,52}

In 1822-1823 (pp. 21-24 of [51]), Ampère examined the force between two circuits, carrying currents I_1 and I_2 , and inferred that this could be written (here in vector notation) as,

$$\mathbf{F}_{\text{on } 1} = \oint_1 \oint_2 d^2 \mathbf{F}_{\text{on } 1}, \quad d^2 \mathbf{F}_{\text{on } 1} = \frac{\mu_0}{4\pi} I_1 I_2 [3(\hat{\mathbf{r}} \cdot d\mathbf{l}_1)(\hat{\mathbf{r}} \cdot d\mathbf{l}_2) - 2 d\mathbf{l}_1 \cdot d\mathbf{l}_2] \frac{\hat{\mathbf{r}}}{r^2} = -d^2 \mathbf{F}_{\text{on } 2}, \quad (65)$$

where $\mathbf{r} = \mathbf{l}_1 - \mathbf{l}_2$ is the distance from a current element $I_2 d\mathbf{l}_2$ at $\mathbf{r}_2 = \mathbf{l}_2$ to element $I_1 d\mathbf{l}_1$ at $\mathbf{r}_1 = \mathbf{l}_1$.^{53,54} The integrand $d^2 \mathbf{F}_{\text{on } 1}$ of eq. (65) has the appeal that it changes sign if elements 1 and 2 are interchanged, and so suggests a force law for current elements that obeys Newton's third law.⁵⁵ However, the integrand does not factorize into a product of terms in the two current elements, in contrast to Newton's gravitational force, and Coulomb's law for the static

⁴⁹The earliest description of eq. (64) as the Biot-Savart force law may be in sec. 2 of [112], and in English, sec. 7-6 of [150]. In France, eq. (64) is often called the Laplace law, although Laplace's (unpublished) contribution seems only to be that the magnetic force on a pole varies as $1/r^2$, as in eq. (62). See p. 339 of [122], p. 23 of [155], p. 15 of [256].

⁵⁰Discussion in English of Ampère's attitudes on the relation between magnetism and mechanics is given in [155, 243, 278]. Historical surveys of 19th-century electrodynamics are given in [114, 256], and studies with emphasis on Ampère include [149, 154, 168, 169, 178, 194, 203, 204, 283]. See also sec. IIA of [260].

⁵¹See p. 166 of [25], and also [178].

⁵²The confirmation that permanent magnetism, due to the magnetic moments of electrons, is Ampèrian (rather than Gilbertian = due to pairs of opposite magnetic charges) came only after detailed studies of positronium (e^+e^- "atoms") in the 1940's [162, 277].

⁵³Ampère sometimes used the notation that the angles between $d\mathbf{l}_i$ and \mathbf{r} are θ_i , and the angle between the plane of $d\mathbf{l}_1$ and \mathbf{r} and that of $d\mathbf{l}_2$ and \mathbf{r} is ω . Then, $d\mathbf{l}_1 \cdot d\mathbf{l}_2 = dl_1 dl_2 (\sin \theta_1 \sin \theta_2 \cos \omega + \cos \theta_1 \cos \theta_2)$, and the force element of eq. (65) can be written as,

$$d^2 \mathbf{F}_{\text{on } 1} = \frac{\mu_0}{4\pi} I_1 I_2 dl_1 dl_2 (\cos \theta_1 \cos \theta_2 - 2 \sin \theta_1 \sin \theta_2 \cos \omega) \frac{\hat{\mathbf{r}}}{r^2} = -d^2 \mathbf{F}_{\text{on } 2}. \quad (66)$$

⁵⁴Ampère also noted the equivalents to,

$$d\mathbf{l}_1 = \frac{\partial \mathbf{r}}{\partial l_1} dl_1, \quad \mathbf{r} \cdot d\mathbf{l}_1 = \mathbf{r} \cdot \frac{\partial \mathbf{r}}{\partial l_1} dl_1 = \frac{1}{2} \frac{\partial r^2}{\partial l_1} dl_1 = r \frac{\partial r}{\partial l_1} dl_1, \quad d\mathbf{l}_2 = -\frac{\partial \mathbf{r}}{\partial l_2} dl_2, \quad \mathbf{r} \cdot d\mathbf{l}_2 = -r \frac{\partial r}{\partial l_2} dl_2, \quad (67)$$

where l_1 and l_2 measure distance along the corresponding circuits in the directions of their currents. Then,

$$d\mathbf{l}_1 \cdot d\mathbf{l}_2 = -d\mathbf{l}_1 \cdot \frac{\partial \mathbf{r}}{\partial l_2} dl_2 = -\frac{\partial}{\partial l_2} (\mathbf{r} \cdot d\mathbf{l}_1) dl_2 = -\frac{\partial}{\partial l_2} \left(r \frac{\partial r}{\partial l_1} \right) dl_1 dl_2 = -\left(\frac{\partial r}{\partial l_1} \frac{\partial r}{\partial l_2} + r \frac{\partial^2 r}{\partial l_1 \partial l_2} \right) dl_1 dl_2, \quad (68)$$

and eq. (65) can also be written in forms closer to those used by Ampère,

$$d^2 \mathbf{F}_{\text{on } 1} = \frac{\mu_0}{4\pi} I_1 I_2 dl_1 dl_2 \left[2r \frac{\partial^2 r}{\partial l_1 \partial l_2} - \frac{\partial r}{\partial l_1} \frac{\partial r}{\partial l_2} \right] \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi} 2I_1 I_2 dl_1 dl_2 \frac{\partial^2 \sqrt{r}}{\partial l_1 \partial l_2} \frac{\hat{\mathbf{r}}}{\sqrt{r}} = -d^2 \mathbf{F}_{\text{on } 2}. \quad (69)$$

⁵⁵In view of this, Maxwell called Ampère the "Newton of electricity" in Art. 528 of [90].

force between electric charges (and between static magnetic poles, whose existence Ampère doubted). As such, Ampère (correctly) hesitated to interpret the integrand as providing the force law between a pair of isolated current elements, *i.e.*, a pair of moving electric charges.⁵⁶

An important qualitative consequence of eq. (65) is that parallel currents attract, and opposite current repel.

Around 1825, Ampère noted, p. 214 of [48], p. 29 of [51], p. 366 of the English translation in [278], that for a closed circuit, eq. (65) can be rewritten as,⁵⁷

$$\mathbf{F}_{\text{on } 1} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{(d\mathbf{l}_1 \cdot \hat{\mathbf{r}}) d\mathbf{l}_2 - (d\mathbf{l}_1 \cdot d\mathbf{l}_2) \hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi} \oint_1 \oint_2 I_1 d\mathbf{l}_1 \times \frac{I_2 d\mathbf{l}_2 \times \hat{\mathbf{r}}}{r^2}, \quad (71)$$

in vector notation (which, of course, he did not use). Ampère made very little comment on this result,⁵⁸ nor factorized it into the forms now associated with the Biot-Savart law(s) (63)-(64).

In 1826, Ampère gave lectures that included discussion of the force on a magnetic pole due to an electric current, noting that the line integral of the tangential force around a closed loop is proportional to the electric current that passes through the loop, independent of the shape of the loop [163, 268]. This was a statement of what is now called “Ampère’s (circuital) law”.⁵⁹

Ampère performed an experiment in 1821-22 [32, 41, 57] that showed a weak effect of electromagnetic induction, which was largely disregarded at the time.⁶⁰

A.14 Ohm

In 1827, Ohm published a treatise [52] containing his famous law, in a form closer to,

$$\mathbf{J} = \sigma \mathbf{E}, \quad (72)$$

where \mathbf{J} is the electric current density and \mathbf{E} is the electric field, both inside the rest frame of a medium with electrical conductivity σ , than to more familiar form, $V = IR$, where V is the potential difference across an electrical resistance R that carries electric current I .

⁵⁶If we follow Ampère in defining a “current element” as being electrically neutral, which is a good (but not exact [269]) approximation for currents in electrical circuits, then a moving charge is not a “current element”, and such elements cannot exist except in closed circuits (contrary to remarks such as in [226]).

⁵⁷Note that for a fixed point 2, $d\mathbf{l}_1 = d\mathbf{r}$, and $dr = d\mathbf{r} \cdot \hat{\mathbf{r}} = d\mathbf{l}_1 \cdot \hat{\mathbf{r}}$. Then, for any function $f(r)$, $df = (df/dr) dr = (df/dr) d\mathbf{l}_1 \cdot \hat{\mathbf{r}}$. In particular, for $f = -1/r$, $df = d\mathbf{l}_1 \cdot \hat{\mathbf{r}}/r^2$, so the first term of the first form of eq. (71) is a perfect differential with respect to \mathbf{l}_1 . Hence, when integrating around a closed loop 1, the first term does not contribute, and it is sufficient to write,

$$\mathbf{F}_{\text{on } 1} = -2 \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r^2} \hat{\mathbf{r}} = -\mathbf{F}_{\text{on } 2} = -\nabla U \quad \text{with} \quad U = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r}, \quad (70)$$

where the magnetic interaction energy U was first given by Neumann, p. 8 of [63].

⁵⁸As a consequence, the form (71) is generally attributed to Grassmann [62], as in [260], for example. See also sec. A.19 below.

⁵⁹This “law” was noted by Maxwell on p. 56 of [77], who deduced from it that $\nabla \times \mathbf{H} = \mathbf{J}$. See also [76], and Art. 498 of [90].

⁶⁰Reviews of this experiment include [106, 154, 190, 194].

Ohm did not define a conductor so much as provide a model for it, with a flavor that electric current is related to the motion of particles. This view became characteristic of the German school in the mid 1800's, but was not taken up by the English or French until much later.

A.15 Green

In 1828, Green published (at his own expense) an essay [53] that introduced the term **potential** for the scalar function,

$$V = \int \frac{\rho}{r} d\text{Vol}, \quad (73)$$

where ρ is the density of electric charge (or of magnetic poles), and r is the distance from the charge to the observation point.

A.16 Faraday

A.16.1 The Electro-tonic State

In secs. 60-61 of his first published series [55] (1831) of *Experimental Researches on Electricity*, Faraday remarked:

Whilst the wire is subject to either volta-electric or magneto-electric induction it appears to be in a peculiar state...the **electro-tonic** state. This peculiar condition shows no known electrical effects whilst it continues; nor have I yet been able to discover any peculiar powers exerted, or properties possessed, by matter whilst retained in this state.

add later quote. relate to momentum?

A.16.2 Electrical Induction *vs.* Action at a Distance

In 1837, sec. 1165 of [59], Faraday's comments on his studies of electrolysis included:

At present I believe ordinary induction in all cases to be an action of contiguous particles consisting in a species of polarity, instead of being an action of either particles or masses at sensible distances;

The notion of lines of force, with tension along them and repulsion between them, appears in sec. 1297:

The direct inductive force, which may be conceived to be exerted in lines between the two limiting and charged conducting surfaces, is accompanied by a lateral or transverse force equivalent to a dilatation or repulsion of these representative lines (1224.); or the attractive force which exists amongst the particles of the dielectric in the direction of the induction is accompanied by a repulsive or a diverging force in the transverse direction (1304.).

His summary in sec. 1304 includes the statements:

I have used the phrases lines of inductive force and curved lines of force (1231. 1297. 1298. 1302.) in a general sense only, just as we speak of the lines of magnetic-force. The lines are imaginary, and the force in any part of them is of course the resultant of compound forces, every molecule being related to every other molecule in all directions by the tension and reaction of those which are contiguous.

A.16.3 Lines of Force in the Magnetic Field

Faraday used the term “lines of magnetic force” in a footnote to sec. 114 of [55] (1831), which “lines” were “depicted by iron filings”, perhaps inspired by Gilbert (sec. A.2) and Descartes (sec. A.3). At this time, Faraday seems to have regarded these “lines” as an aspect of the iron filings, rather than as an independent phenomenon.

In 1845, sec. 2147 of [66], the term “magnetic field” appears for the first time in print:

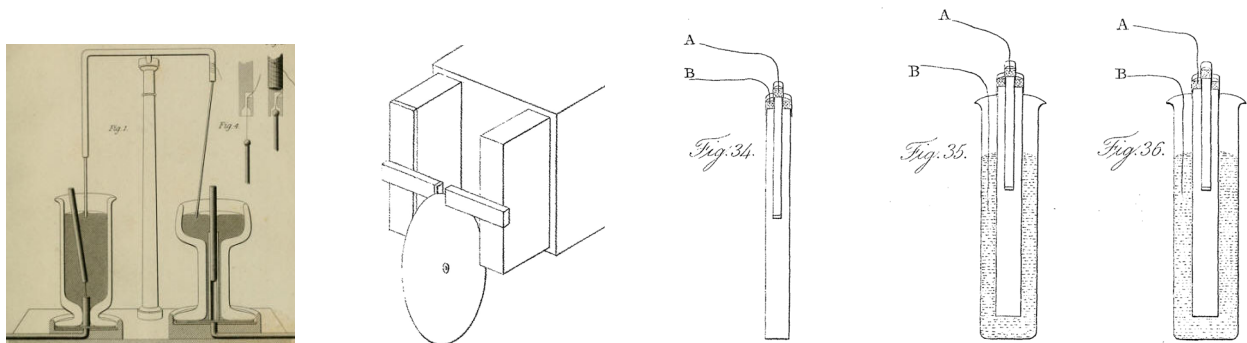
The ends of these bars form the opposite poles of contrary name; the magnetic field between them can be made of greater or smaller extent, and the intensity of the lines of magnetic force be proportionately varied.

It seems to this author that Faraday used the term “magnetic field” to mean a region where magnetic lines of force or significant (in the sense of the sporting term “playing field”), rather than that the magnetic field is a measure of the density of the lines of force (corresponding to our present vector field \mathbf{B}). That is, Faraday coined the term “magnetic field”, but did not quite give this term its contemporary meaning, in which the electromagnetic fields contain energy, momentum and angular momentum.

While the notion of electric and magnetic fields is often stated as due to Faraday, it is perhaps better to think of his vision of electric and magnetic lines of force that have physical meaning even in “vacuum” as an early version of “string theory”.

A.16.4 Motors and Generators

In 1821, Faraday demonstrated a primitive electric motor [40], shown on the left below, in which a current-carrying pencil magnet rotated over the surface of a cone when its lower end was in a pool of mercury, completing the electric circuit.



In 1831, sec. 88 of [55], Faraday gave his first demonstration of an electrical generator/dynamo, in which a copper disk that rotated in an external magnetic field generated a current in a circuit with a sliding contact with the disk, sketched in the middle figure above.

In 1832, secs. 217-227 of [56], Faraday replaced the copper disk and the external magnet by a conducting, rotating magnet, whose self field acting on the “free” charges in the magnet also produced a current in the circuit, which effect is often called a homopolar generator.

A.16.5 Electromagnetism and Light

Airy 1846 ray vibrations

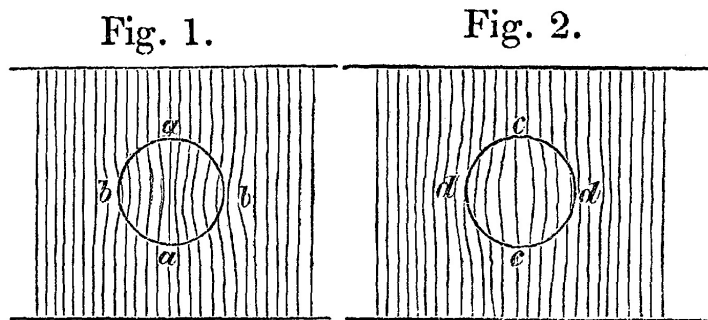
A.16.6 Interactions with W. Thomson

As will be noted further in sec. A.22 below, Faraday interacted with the young W. Thomson from 1845 onwards, when the latter began a program (completed by Maxwell, sec. A.23) to give a mathematical form to Faraday's concept of lines of force. Hence, the evolution in this period of Faraday's vision of lines of force may have been influenced by Thomson.

In 1850, sec. 2806 of [72], Faraday wrote:

Any portion of space traversed by lines of magnetic power, may be taken as such a (magnetic) field, and there is probably no space without them. The condition of the field may vary in intensity of power, from place to place, either along the lines or across them...

2807. When a paramagnetic conductor, as for instance, a sphere of oxygen, is introduced into such a magnetic field, considered previously as free from matter, it will cause a concentration of the lines of force on and through it, so that the space occupied by it transmits more magnetic power than before (fig. 1). If, on the other hand, a sphere of diamagnetic matter be placed in a similar field, it will cause a divergence or opening out of the lines in the equatorial direction (fig. 2); and less magnetic power will be transmitted through the space it occupies than if it were away.



Here, one can identify Faraday's usage of the term "magnetic power" with the magnetic flux $\Phi = \int \mathbf{B} \cdot d\mathbf{Area}$, although Faraday does not extrapolate this usage to a notion of a locally defined field strength \mathbf{B} .

A further consequence of his interaction with Thomson appears to be that in 1852, beginning in sec. 3070 of [73], Faraday wrote about "lines of force" more abstractly, but without full commitment to their physical existence independent of matter. Thus, in sec. 3075 he stated:

I desire to restrict the meaning of the term line of force, so that it shall imply no more than the condition of the force in any given place, as to strength and direction; and not to include (at present) any idea of the nature of the physical cause of the phenomena...

A few sentences later he continued:

...for my own part, considering the relation of a vacuum to the magnetic force and the general character of magnetic phenomena external to the magnet, I am more inclined to the notion that in the transmission of the force there is such an action, external to the magnet, than that the effects are merely attraction and repulsion at a distance. Such an action may be a function of the ether; for it is not at all unlikely that, if there be an ether, it should have other uses than simply the conveyance of radiations (2591. 2787.).

In sec. 3175, at the end of [73], he added:

...wherever the expression line of force is taken simply to represent the disposition of the forces, it shall have the fullness of that meaning; but that wherever it may seem to represent the idea of the physical mode of transmission of the force, it expresses in that respect the opinion to which I incline at present.

This has led many to infer that Faraday then believed in the physical existence of the lines of force even though he could not “prove” that.

Faraday’s famous notion, that induced electrical currents are associated with wires “cutting” lines of magnetic force, is presented in sec. 3104, and a version of what is now called Faraday’s law,

$$\mathcal{EMF} = -\frac{1}{c} \frac{d}{dt} \int \mathbf{B} \cdot \text{Area}, \quad (74)$$

is given verbally in sec. 3115.⁶¹ In sec. 3117 Faraday noted that magnetic lines of force form closed circuits. In sec. 3118 Faraday (re)affirmed that magnetic field lines do not rotate with a rotating magnet, and performs various experiments with what is now called a unipolar (or homopolar) generator to demonstrate this, which experiments are an early investigation of the relativity of rotating frames.

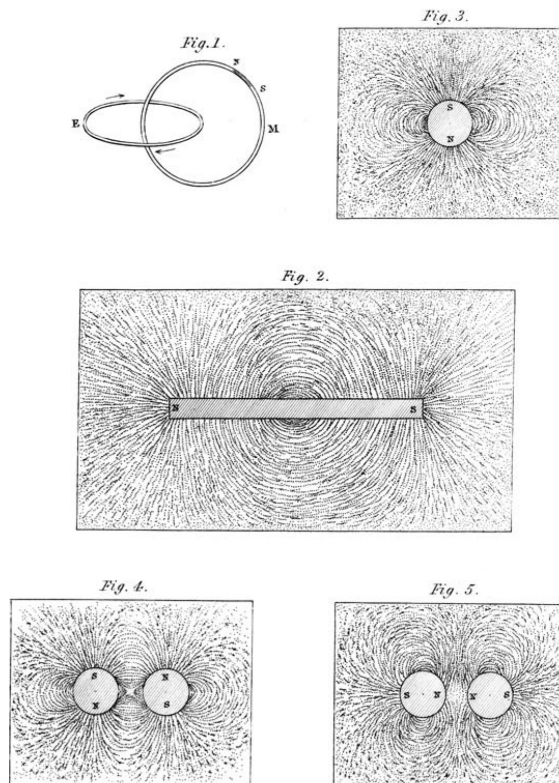
In 1852, Faraday also published a set of more speculative comments [74] in the *Phil. Mag.* (rather than *Phil. Trans. Roy. Soc. London*, the usual venue for his *Experimental Researches*), arguing more strongly for the physical reality of the lines of force.

In sec. 3258 he considered the effect of a magnet in vacuum, concluding (perhaps for the first time) that the lines of force have existence independent of a material medium:

A magnet placed in the middle of the best vacuum we can produce...acts as well upon a needle as if it were surrounded by air, water or glass; and therefore these lines exist in such a vacuum as well as where there is matter.

Faraday used examples of magnets and iron filings in various configurations to reinforce his vision of a tension along the lines of forces, and in sec. 3295 added the insight that there is a lateral repulsion between adjacent lines, referring to Fig. 5 below.

⁶¹One should not infer from this that Faraday had an explicit notion of the magnetic field \mathbf{B} as a measure of the density of lines of magnetic force. Rather, he emphasized the total number of lines within some area (the magnetic flux) as the “amount of magnetic force” (sec. 3109).



Faraday's last published comments on lines of force are in [75].

A.17 Gauss

In 1867 Gauss published an analysis that he dated to 1835 (p. 609 of [58]), in which he states that a time-dependent electric current leads to an electric field which is the time derivative of what we now called the vector potential. English translation from [233]:

The Law of Induction

Found out Jan. 23, 1835, at 7 a.m. before getting up.

1. The electricity producing power, which is caused in a point P by a current-element γ , at a distance from P , $= r$, is during the time dt the difference in the values of γ/r corresponding to the moments t and dt , divided by dt . where γ is considered both with respect to size and direction. This can be expressed briefly and clearly by,

$$-\frac{d(\gamma/r)}{dt}. \quad (75)$$

On p. 612 (presumably also from 1835), Gauss notes a relation (here transcribed into vector notation) between the vector $\mathbf{A} = \oint d\mathbf{l}/r$ and the magnetic scalar potential Ω of a circuit with unit electrical current (which he relates to the solid angle subtended by the circuit on p. 611),

$$-\nabla\Omega = \nabla \times \mathbf{A}. \quad (76)$$

While we would identify eq. (76) with the magnetic field \mathbf{H} , Gauss called it the “electricity-generating force”.

In any case, eq. (76) is the earliest appearance of the curl operator (although published later than MacCullagh's (sec. A.18) use of this.

A.18 MacCullagh

A.19 Grassmann

In 1845, Grassmann [62] noted that although Ampère claimed [51] that his force law was *uniquement déduite de l'expérience*, it included the assumption that it obeyed Newton's third law. He noted that Ampère's law (65) implies that the force is zero for parallel current elements whose line of centers makes angle $\cos^{-1} \sqrt{2/3}$ to the direction of the currents, which seemed implausible to him. Grassmann claimed that, unlike Ampère, he would make no "arbitrary" assumptions, but in effect he assumed that there is no magnetic force between collinear current elements, which leads to a force law,

$$\mathbf{F}_{\text{on } 1} = \oint_1 \oint_2 d^2 \mathbf{F}_{\text{on } 1}, \quad d^2 \mathbf{F}_{\text{on } 1} = I_1 d\mathbf{l}_1 \times \frac{I_2 d\mathbf{l}_2 \times \hat{\mathbf{r}}}{c^2 r^2}, \quad (77)$$

in vector notation (which, of course, Grassmann did not use). While $d^2 \mathbf{F}_{\text{on } 1}$ is not equal and opposite to $d^2 \mathbf{F}_{\text{on } 2}$, Grassmann showed that the total force on circuit 1 is equal and opposite to that on circuit 2, $\mathbf{F}_{\text{on } 1} = -\mathbf{F}_{\text{on } 2}$.

Grassmann's result is now called the Biot-Savart force law,

$$\mathbf{F}_{\text{on } 1} = \oint_1 \frac{I_1 d\mathbf{l}_1 \times \mathbf{B}_2}{c}, \quad \mathbf{B}_2 = \oint_2 \frac{I_2 d\mathbf{l}_2 \times \hat{\mathbf{r}}}{cr^2}, \quad (78)$$

although Grassmann did not identify the quantity \mathbf{B}_2 as the magnetic field.

A.20 Neumann

In 1845, Neumann [63] independently arrived at (what is now written as) the form (77), and verified that it gives the same total force between closed circuits as does Ampère's eq. (65).

In sec. 11 of [63], Neumann also discussed the magnetic energy of two circuits, calling this the "potential",⁶²

$$U = \oint_1 \oint_2 \frac{I_1 d\mathbf{l}_1 \cdot I_2 d\mathbf{l}_2}{c^2 r}. \quad (79)$$

We now also write this as,

$$U = \oint_i \frac{I_i d\mathbf{l}_i \cdot \mathbf{A}_j}{c}, \quad \mathbf{A}_j = \oint_j \frac{I_j d\mathbf{l}_j}{cr}, \quad (80)$$

such that Neumann is often credited in inventing the vector potential \mathbf{A} , although he appears not to have factorized his eq. (79) into eq. (80).

⁶²If we write eq. (79) as $U = I_1 I_2 M_{12}$, then M_{12} is the mutual inductance of circuits 1 and 2. Neumann included a factor of 1/2 in his version of eq. (79), associated with his choice of units.

A.21 Weber

Weber was perhaps the last major physicist who did not use electric and magnetic fields to describe electromagnetism, preferring an (instantaneous) action-at-a-distance formulation for the forces between charges (1846 [67], p. 144 of [104]),⁶³

$$F = \frac{ee'}{r^2} \left[1 - a^2 \left(\frac{dr}{dt} \right)^2 + 2a^2 r \frac{d^2 r}{dt^2} \right]. \quad (81)$$

This was the first published force law for moving charges (which topic Ampère refused to speculate upon). The constant a has dimensions of velocity⁻¹, and was later (1856) written by Weber and Kohlrausch [79] as $1/C$, who noted that their C is the ratio of the magnetic units to electrical units in the description of static phenomenon, which they determined experimentally to have a value close to 4.4×10^8 m/s. Apparently, they regarded it as a coincidence that their C is roughly $\sqrt{2}$ times the speed c of light.

Weber’s alternative vector potential.....

A.22 W. Thomson (Lord Kelvin)

A.22.1 Force Fields

In 1842 (at age 18!), W. Thomson [61] noted an analogy between the (vector) flow of heat and the “attractive force” of electricity. At that time he was concerned with electrostatics, for which it is natural to consider the force only at the locations of charges and not in the space between them. In contrast, the flow of heat exists in the space between sources and sinks of heat, so Thomson’s analogy perhaps started him thinking about possible significance of electrical forces away from the location of electric charges.

Thomson appears to have become aware of Faraday’s work in 1845, and soon published initial comments [64] about transcribing Faraday’s notions into mathematical form. He noted the contrast between Coulomb’s action-at-distance view of electrical forces, and Faraday’s view (reminiscent of Descartes’s) that these forces are transmitted via some kind of “action of contiguous particles of some intervening medium”, and proceeded to argue that these are what might now be called “dual” explanations of electricity. We see in this discussion the beginning of Thomson’s lifelong vision of a mechanical ether supporting electricity and magnetism.⁶⁴

In 1846, Thomson [68], p. 63, described the electrical force due to a unit charge at the origin “exerted at the point (x, y, z) ” as \mathbf{r}/r^3 , without explicit statement that a charge exists at the point to experience the force. In the view of this author, that statement is the first mathematical appearance of the electric field in the literature, although neither vector notation nor the term “electric field” were used by Thomson.

⁶³For an extensive discussion of Weber’s electrodynamics, see [240]. Maxwell gave a review of the German school of electrodynamics of the mid 19th century in the final chapter 23 of his *Treatise* [90].

⁶⁴“I never satisfy myself until I can make a mechanical model of a thing. If I can make a mechanical model I can understand it. As long as I cannot make a mechanical model all the way through I cannot understand; and that is why I cannot get the electro-magnetic theory,” p. 270 of [98].

He immediately continued with the example of a “point” magnetic dipole \mathbf{m} , whose scalar potential is $\Phi = \mathbf{m} \cdot \mathbf{r}/r^3$, noting that the magnetic force $-\nabla\Phi$ on a unit magnetic pole p can also be written as $\nabla \times \mathbf{A}$ where $\mathbf{A} = \mathbf{m} \times \mathbf{r}/r^3$ (although Thomson did not assign a symbol to the vector \mathbf{A}). This discussion is noteworthy for the sudden appearance of the vector potential of a magnetic dipole (with no reference to Neumann, whose 1845 paper [63] implied this result, but was not explicit about its application to Thomson’s example).

In a major paper on magnetism in 1849 [70], Thomson still did not use the term “field”, but wrote in sec. 48:

The resultant force at a point in space, void of magnetized matter, is the force that the north pole of a unit-bar (or a positive unit of imaginary magnetic matter), if placed at this point, would experience.

The term “magnetic field” in the contemporary sense first appears in 1851 on p. 179 of [71], where Thomson wrote:

Definition.—Any space at every point of which there is a finite magnetic force is called “a field of magnetic force;” or, *magnetic* being understood, simply “a field of force;” or, sometimes, “a magnetic field.”

Definition.—A “line of force” is a line drawn through a magnetic field in the direction of the force at each point through which it passes; or a line touched at each point of itself by the direction of the magnetic force.

A.22.2 B and H

In sec. 78 of [70], Thomson considered the magnetic-field vector (X, Y, Z) that we now identify with \mathbf{B} . In considerations of the effect of the magnetic field on hypothetical magnetic poles inside small cavities in a medium with magnetization density \mathbf{M} ,

A.23 Maxwell

A.23.1 Electromagnetic Momentum

In Maxwell’s earliest publication (at age 24) on electromagnetism [77], Part II is titled *On Faraday’s “Electro-tonic State”*. On p. 52 Maxwell says:

Considerations of this kind led Professor Faraday to connect with his discovery of the induction of electric currents the conception of a state in which all bodies are thrown by the presence of magnets and currents. This state does not manifest itself by any known phenomena as long as it is undisturbed, but any change in this state is indicated by a current or tendency towards a current. To this state he gave the name of the “Electro-tonic State”.

Then on p. 65 Maxwell gives his theory of the electro-tonic state:

The entire electro-tonic intensity round the boundary of an element of surface measures the quantity of magnetic induction which passes through that surface, or, in other words, the number of lines of magnetic force which pass through the surface.

In vector notation, with \mathbf{A} as the electro-tonic intensity and \mathbf{B} as the magnetic induction,

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{Area}. \quad (82)$$

Thus, we recognize Maxwell's electro-tonic intensity as the vector potential,⁶⁵ and it is often said that Faraday's electro-tonic state is the vector potential.

Maxwell continued discussion of his vector \mathbf{A} on p. 290 of [83], now calling it the “electrotonic state” and seeking a mechanical interpretation in terms of “molecular vortices”. On p. 339 he associates the vector \mathbf{A} with a kind of “reduced momentum”, arguing that if \mathbf{A} is changing there is a force on a unit electric charge given by $d\mathbf{A}/dt$.^{66,67}

In his great paper of 1865 [85], Maxwell reinforced this interpretation of his electro-tonic intensity/electrotonic state, now calling it the “electromagnetic momentum”. His discussion in sec. 57, p. 481 concerns the mechanical momentum \mathbf{P}_{mech} of a unit charge q subject to the electric field induced by a changing \mathbf{A} ,

$$\frac{d\mathbf{P}_{\text{mech}}}{dt} = \mathbf{F} = q\mathbf{E} = -\frac{q}{c} \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{P}_{\text{mech}} = \mathbf{P}_0 - \frac{q\mathbf{A}}{c}. \quad (83)$$

A charge q that somehow arrives at a point where the vector potential is \mathbf{A} , it will have extracted momentum $-q\mathbf{A}/c$ from the electromagnetic field. Supposing that all charges (and possible other masses) started from rest, with zero total initial momentum, for momentum to be conserved it must be that the field now stores momentum $q\mathbf{A}/c$, leading to the interpretation that \mathbf{A} is a kind of electromagnetic momentum.

The total electromagnetic momentum stored in the field must be,

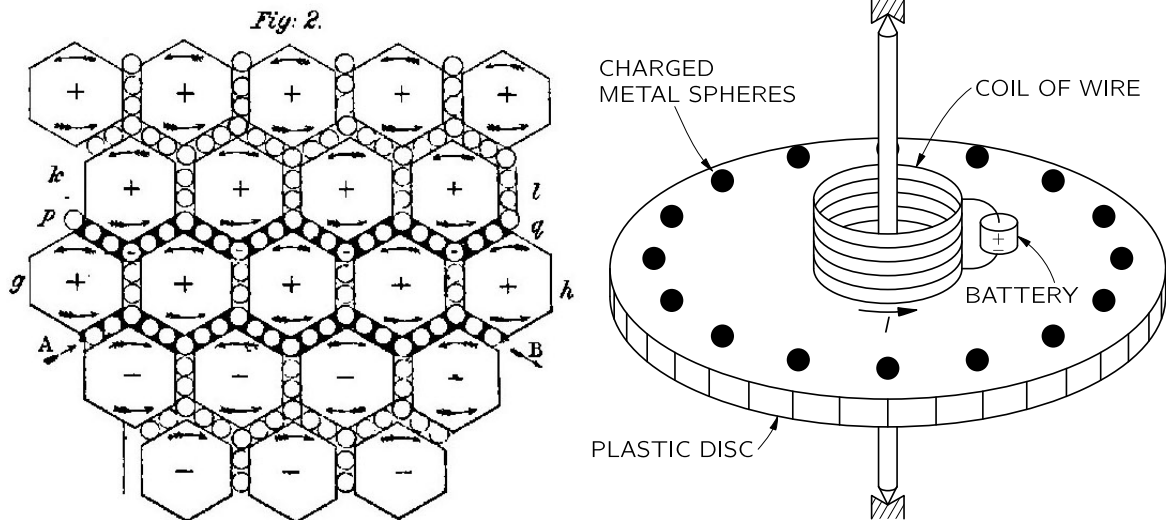
$$\mathbf{P}_{\text{EM}}^{(\text{M})} = \sum \frac{q_i \mathbf{A}_i}{c} \rightarrow \int \frac{\rho \mathbf{A}}{c} d\text{Vol}, \quad (84)$$

where the superscript M indicates that this form is due to Maxwell.

⁶⁵Maxwell noted on p. 59 of [77] that $\mathbf{B} = \nabla \times \mathbf{A}$, and that \mathbf{A} can be subject to what is now called a gauge transformation while leaving \mathbf{B} unchanged, so he is free to set $\nabla \cdot \mathbf{A} = 0$, this being the first appearance of the Coulomb gauge. On p. 62, Maxwell attributed an energy $\int \mathbf{J} \cdot \mathbf{A} d\text{Vol}$ to the interaction of a current density with the electro-tonic state, and on p. 64 he remarked that a changing \mathbf{A} leads to an electric field $\mathbf{E} = -(1/c)\partial\mathbf{A}/\partial t$ (in Gaussian units, which we employ in this note), where c is the speed of light in vacuum. On p. 73 he gives a sample computation of \mathbf{A} for a sphere of radius a with a $\sin\theta$ winding, finding that $A_\phi = (rB_0/2)\sin\theta$ for $r < a$ and $(a^3B_0/2r^2)\sin\theta$ for $r > a$, where B_0 is the uniform magnetic field inside the sphere.

⁶⁶Maxwell appears to have reversed the sign of \mathbf{A} in [83] compared to the now-usual convention.

⁶⁷Maxwell discusses the force on a unit charge on p. 342 of [83], giving in eq. (77) what is now called the Lorentz force law.



A.23.2 Electromagnetic Field Energy and Stress Tensor

A.23.3 Radiation Pressure

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