Pulling a Mass

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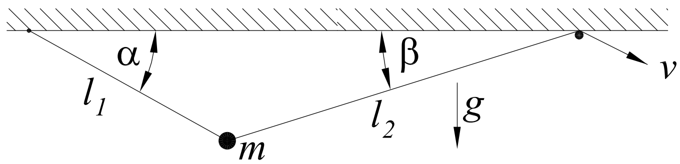
1 Problem

Mass m is suspended from a fixed point A by a string of fixed length l_1 that makes angle α to the horizontal, while also pulled at constant speed v by a second string that passes over a fixed pulley. What is is the tension T in the second string when that string makes angle β to the horizontal, as in the sketch below?



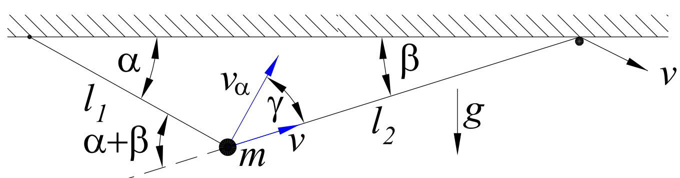
2 Solution

This problem is something of a "trick" question as a simpler answer can be given in terms of two angles α and β, although there is only one independent angle, say, α. It was posed as Prob. 21 on p. 188 of [1], with a solution by Newtonian methods given on p. 208. Here we give a solution via Lagrange's method, which requires use of a Lagrange multiplier as there exists a velocity-dependent (nonholonomous) constraint.

The velocity constraint can usefully be expressed in terms of angles α and β by noting that the azimuthal velocity $v_\alpha = -l_1\dot\alpha = -l_1 d\alpha/dt$ of mass m is related the the velocity v of mass m along the direction of string 2 by

$$v = v_\alpha \cos\gamma = v_\alpha \sin(\alpha+\beta) = -l_1\dot\alpha \sin(\alpha+\beta), \tag{1}$$

since angle $\gamma = \pi/2 - (\alpha+\beta)$, referring to the figure below.



The prescription[1] to include the effect of the constraint (1) in Lagrange's equation is that

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\alpha} - \frac{\partial \mathcal{L}}{\partial \alpha} = -l_1 \sin(\alpha+\beta)\lambda, \tag{2}$$

where λ is the Lagrange multiplier, with the physical significance of the tension T in string 2. Note that $Tl_1\sin(\alpha+\beta)$ is the torque due to tension T about the support point of string 1, so the right side of eq. (2) is the generalized force corresponding to angle α.

[1]Notes by the author on the method of Lagrange multiplier are around p. 68 of [2].

In the present problem, the Lagrangian $\mathcal{L}$ based on independent angle α is

$$\mathcal{L} = \text{KE} - \text{PE} = \frac{ml_1^2\dot{\alpha}^2}{2} + mgl_1 \sin\alpha, \tag{3}$$

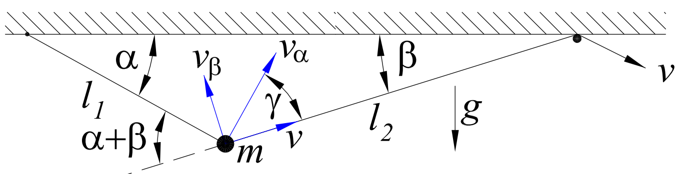
so eq. (2) implies that

$$ml_1^2\ddot{\alpha} - mgl_1\cos\alpha = -l_1\sin(\alpha+\beta)T, \qquad T = \frac{mg\cos\alpha}{\sin(\alpha+\beta)} - \frac{ml_1\ddot{\alpha}}{\sin(\alpha+\beta)}\,. \tag{4}$$

A relation for the acceleration $a_\alpha = -l_1\ddot{\alpha}$ of mass m with respect to the fixed point of string 1 can be found by first noting that the radial acceleration (with respect to the fixed point of string 1) has magnitude

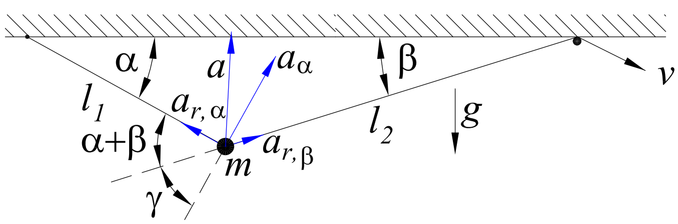
$$a_{r,\alpha} = \frac{v_\alpha^2}{r} = \frac{v^2}{l_1\sin^2(\alpha+\beta)}\,. \tag{5}$$

A subtle trick is to consider also the radial acceleration $a_{r,\beta} = v_\beta^2/l_2$ of mass m with respect to the fixed pulley, as sketched in the two figures below.



We have that $v_\beta = v_\alpha\sin\gamma = v_\alpha\cos(\alpha+\beta)$, and hence the magnitude of the radial acceleration with respect to the pulley is

$$a_{r,\beta} = \frac{v_\beta^2}{l_2} = \frac{v_\alpha^2\cos^2(\alpha+\beta)}{l_2} = a_\alpha\cos\gamma - a_{r,\alpha}\cos(\alpha+\beta) = a_\alpha\sin(\alpha+\beta) - a_{r,\alpha}\cos(\alpha+\beta). \tag{6}$$



From eq. (6) we have, recalling eqs. (1) and (5) and noting that $l_1/\sin\beta = l_2/\sin\alpha$,

$$\begin{aligned} a_\alpha = -l_1\ddot{\alpha} = \frac{a_{r,\beta}}{\sin(\alpha+\beta)} + \frac{a_{r,\alpha}\cos(\alpha+\beta)}{\sin(\alpha+\beta)} &= \frac{v_\alpha^2\cos^2(\alpha+\beta)}{l_2\sin(\alpha+\beta)} + \frac{v^2\cos(\alpha+\beta)}{l_1\sin^2(\alpha+\beta)} \\ &= \frac{v^2\cos^2(\alpha+\beta)}{l_2\sin^3(\alpha+\beta)} + \frac{v^2\cos(\alpha+\beta)}{l_1\sin^2(\alpha+\beta)}\,. \end{aligned} \tag{7}$$

Finally, from eq. (4), the tension in string 2 is[2]

$$T = \frac{mg\cos\alpha}{\sin(\alpha+\beta)} + \frac{mv^2\cot(\alpha+\beta)}{\sin^2(\alpha+\beta)}\left(\frac{\cot(\alpha+\beta)}{l_2} + \frac{1}{l_1}\right). \tag{8}$$

[2] There is a small error in the expression for T on p. 209 of [1].

We can use eq. (8) for the tension T in the first of eq. (4) to obtain an equation of motion involving both angles α and β, Since angle β is a known function of angle α, we could display a (complicated) equation of motion in terms of angle α only.

The author was introduced to this problem by Clement Chan.

References

[1] J. Wang and B. Ricardo, *Competitive Physics: Mechanics and Waves*, (World Scientific, 2019). http://kirkmcd.princeton.edu/examples/mechanics/wang_19.pdf

[2] K.T. McDonald, *Physics 205 Lecture 6*. http://kirkmcd.princeton.edu/examples/Ph205/ph205l6.pdf