Energy Conservation
in a Pulley + Mass System

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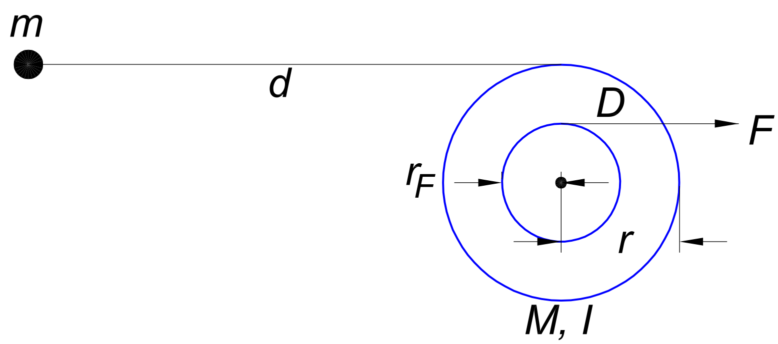
1 Problem

Discuss the motion in the frictionless system of a pulley of mass M, and moment of inertia I about its axis, when constant force F is applied via a string at radius r_F from the axis of pulley, while a mass m is also connected to the pulley via a string at radius r. Verify that the kinetic energy of the system, after starting from rest, equals the work done by force F.

The translational motion is entirely in the direction of the strings, as sketched below.



This problem was suggested by Mario Carvajal.

2 Solution

2.1 Massless Pulley with Fixed Axis

We first consider a simpler case, of a massless pulley with fixed axis.

The massless pulley has zero moment of inertia I, so the torque equation for the pulley is,

$$\tau = I\alpha = 0 = r_F F - r f, \qquad f = \frac{r_F}{r} F. \tag{1}$$

where f is the (constant) tension in the string that connects mass m to the pulley. Note that $f = F$ only if $r = r_F$.

As the pulley rotates through angle θ, the end of the string supporting force F advances by distance $\Delta x_F = r_F\,\theta$, while mass m is pulled towards the pulley by distance $\Delta x_m = r\,\theta$. That is,

$$\theta = \frac{\Delta x_F}{r_F} = \frac{\Delta x_m}{r}\,, \qquad \Delta x_m = \frac{r}{r_F}\Delta x_F. \tag{2}$$

Supposing force F is first applied at time $t = 0$, when mass m is at rest, the latter accelerates according to,

$$a_m = \frac{f}{m}, \qquad v_m = \frac{f}{m} t, \qquad \Delta x_m = \frac{f}{2\,m} t^2. \tag{3}$$

For completeness, we note that the end of the string that supports force F moves according to,

$$\Delta x_F = \frac{r_F}{r} \Delta x_m = \frac{r_F}{r} \frac{f}{2\,m} t^2 = \frac{r_F^2}{r^2} \frac{F}{2\,m} t^2, \qquad v_F = \frac{r_F^2}{r^2} \frac{F}{m} t, \qquad a_F = \frac{r_F^2}{r^2} \frac{F}{m}. \tag{4}$$

The kinetic energy of mass m at time t is, using eqs. (1)-(3),

$$\mathrm{KE}_m = \frac{m\,v_m^2}{2} = \frac{m}{2} \frac{f^2}{m^2} t^2 = f \frac{f}{2\,m} t^2 = f \Delta x_m = \frac{r_F}{r} F \Delta x_m = F \Delta x_F = W_F, \tag{5}$$

such that the work $W_F = F \Delta x_F$ done by force F equals the (change of) kinetic energy of mass m.

2.2 Massive Pulley with Fixed Axis

In this case, the torque equation for the pulley becomes,

$$\tau = I\alpha = I\ddot{\theta} = r_F F - r f, \qquad \omega = \dot{\theta} = \frac{r_F F - r f}{I} t, \qquad \theta = \frac{r_F F - r f}{2I} t^2. \tag{6}$$

Equations (2)-(3) remain the same, while the kinetic energy of the system includes the rotational energy of the pulley,

$$\begin{aligned} \mathrm{KE} &= \frac{m\,v_m^2}{2} + \frac{I\omega^2}{2} = \frac{m}{2} \frac{f^2}{m^2} t^2 + \frac{(r_F F - r f)^2}{2I} t^2 = f \frac{f}{2\,m} t^2 + (r_F F - r f)\,\theta \\ &= f \Delta x_m + F \Delta x_F - f \Delta x_m = F \Delta x_F = W_F. \end{aligned} \tag{7}$$

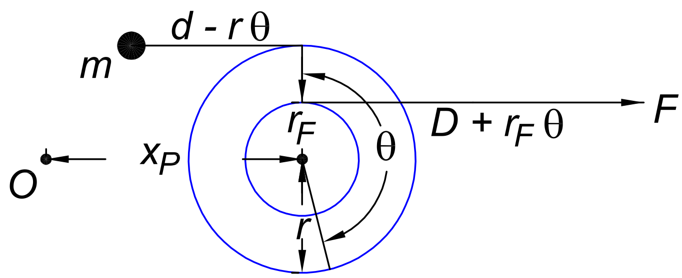
Again, the work done by force F appears as the kinetic energy of the system.

2.3 Massive Pulley whose Axis is not Fixed

We take the center of the pulley to be at the origin at time $t = 0$, when force F is first applied, in the $-x$ direction, and the end of the string is at distance D from the pulley. Also at $t = 0$, the length of the string between the pulley and mass m is d, such that the initial x-coordinate of the mass is $x_{m,0} = -d$.

At a later time t, the center of the pulley is at x_P and the pulley has rotated by angle $\theta > 0$. The string by which force F is applied now has unwrapped by length $r_F \theta$, and its end has coordinate $x_F = x_P + D + r_F\,\theta$. Similarly, the coordinate of mass m at time t is $x_m = x_P - d + r\,\theta$, since length $r\theta$ of its string is now wrapped around the pulley. That is,

$$\Delta x_F = x_P + r_F\,\theta, \qquad \Delta x_m = x_P + r\,\theta. \tag{8}$$



Denoting the tension in the string connected to mass m by $f > 0$, the equations of motion for the mass and pulley are,

$$m\,a = m\,\ddot{x}_m = m(\ddot{x}_P + r\,\ddot{\theta}) = f, \tag{9}$$

$$M\,a_P = M\ddot{x}_P = F - f, \tag{10}$$

$$I\alpha = I\ddot{\theta} = \tau = r_F F - r\,f. \tag{11}$$

These are three equations for the three unknowns x_P, θ and f.

Combining eqs. (9) and (10), we find,

$$f = \frac{m}{M}(F - f) + m\,r\,\ddot{\theta}, \qquad f = \frac{m}{M+m}F + \frac{Mm}{M+m}r\,\ddot{\theta}, \tag{12}$$

and using this in eq. (11) leads to,

$$I\ddot{\theta} = r_F F - \frac{m}{M+m}rF - \frac{Mm}{M+m}r^2\,\ddot{\theta}, \qquad I'\ddot{\theta} = RF, \tag{13}$$

where,

$$I' = I + \frac{Mm\,r^2}{M+m}, \qquad R = r_F - \frac{m}{M+m}r. \tag{14}$$

For $r/r_F > (M+m)/m$, the quantity R is negative, and the motion would not be as in the figure above; instead the pulley would rotate "backwards", and its center would move away from the x-axis.

We restrict further discussion to the case that $R > 0$.

From eq. (13), we have that,

$$\omega \equiv \dot{\theta} = \frac{RF}{I'}t, \qquad \theta = \frac{RF}{2I'}t^2. \tag{15}$$

The tension in the string connected to mass m follows from eqs. (12)-(13) as,

$$f = \frac{m}{M+m}\left(1 + \frac{MrR}{I'}\right)F = \frac{m}{M+m}\frac{I + Mrr_F}{I + \frac{Mmr^2}{M+m}}F, \tag{16}$$

which is a constant force, with $f \le F$ for $R > 0$.

The motion of mass m follows from eq. (9) as,

$$v_m \equiv \dot{x}_m = \frac{f}{m}t, \qquad \Delta x_m = \frac{f}{2\,m}t^2, \tag{17}$$

and the motion of the center of the pulley follows from eq. (10) as,

$$v_P \equiv \dot{x}_P = \frac{F-f}{M}t, \qquad x_P = \frac{F-f}{2M}t^2. \tag{18}$$

The kinetic energy of the system is,

$$\begin{aligned}
\mathrm{KE} &= \frac{m\,v_m^2}{2} + \frac{Mv_P^2}{2} + \frac{I\omega^2}{2} = \frac{f^2t^2}{2\,m} + \frac{(F-f)^2t^2}{2M} + I\frac{R^2F^2t^2}{2I'^2} \\
&= f\Delta x_m + (F-f)x_P + F\frac{IR}{I'}\theta = f(x_p + r\,\theta) + (F-f)x_P + F\frac{IR}{I'}\theta \\
&= Fx_P + f\,r\,\theta + F\frac{IR}{I'}\theta = Fx_P + F\frac{m\,r}{M+m}\left(1 + \frac{MrR}{I'}\right)\theta + F\frac{IR}{I'}\theta \\
&= Fx_P + F(r_F - R)\left(1 + \frac{MrR}{I'}\right)\theta + F\frac{IR}{I'}\theta \\
&= F(x_P + r_F\,\theta) + FR\,\theta\left[r_F\frac{Mr}{I'} - \left(1 + \frac{MrR}{I'}\right) + \frac{I}{I'}\right] \\
&= F\Delta x_F + FR\,\theta\left[(r_F - R)\frac{Mr}{I'} + \frac{I}{I'} - 1\right] \\
&= W_F + FR\,\theta\left[\frac{m}{M+m}\frac{Mr^2}{I'} + \frac{I}{I'} - 1\right] = W_F.
\end{aligned} \tag{19}$$

Once again, the work done by force F appears as the kinetic energy of the system.