

Gauging Away Polarization States of Waves

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1 Problem

Discuss how the concept of gauge invariance can lead to an understanding of how/why electromagnetic waves can have only two independent polarization states.

Comment also on gravitational waves.

2 Solution

2.1 Electromagnetic Waves

A familiar argument considers a unit, plane electromagnetic wave in free space with electric field given by the real part of,

$$\mathbf{E} = \hat{\mathbf{E}}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \quad (1)$$

where $\hat{\mathbf{E}}_0$ is a unit vector (possibly complex), \mathbf{k} is the wave vector and $\omega = kc$ is the angular frequency, with c being the speed of light in vacuum. This field obeys the first Maxwell equation, $\nabla \cdot \mathbf{E} = 0$ (in empty space), which implies that,

$$\hat{\mathbf{E}}_0 \cdot \mathbf{k} = 0, \quad (2)$$

and hence there are only two independent possibilities for the unit vector $\hat{\mathbf{E}}_0$, which are correspond to two independent “polarization” states of the wave, both of which have electric field transverse to the wave vector. Whereas, for waves inside matter, in general $\nabla \cdot \mathbf{E} \neq 0$, so there can be waves with longitudinal, as well as transverse, polarization.

Here, we consider a longer argument based on the scalar and vector potentials ϕ and \mathbf{A} , which has the possible merit of being extendable to the case of gravitational waves.

The potentials can be considered as components of a 4-vector potential,

$$\phi_\mu = (\phi, \mathbf{A}). \quad (3)$$

Plane waves of the potentials have the form,

$$\phi_\mu = \epsilon_\mu e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \quad (4)$$

where ϵ_μ is a constant 4-vector. In principle, there are 4 independent types of polarization for waves of the 4-potential.

Because the electromagnetic fields \mathbf{E} and \mathbf{B} can be deduced from derivatives of the potentials, the latter have some degree of arbitrariness, which fact has come to be discussed under the theme of *gauge invariance*.¹ One consequence of gauge invariance is that one can

¹For a historical review, see [1].

choose to enforce one relation among the derivatives of the potentials, now called a gauge condition, or choice of gauge. When electromagnetic waves are concerned, a particularly useful choice is the Lorenz condition [2],²

$$\partial_\mu \phi^\mu = 0 = \frac{1}{c} \frac{\partial \phi}{\partial t} - \nabla \cdot \mathbf{A} \quad (\text{Lorenz}), \quad (5)$$

in Gaussian units. Applying the Lorenz-gauge condition (5) to the 4-potential wave (4), we have that,

$$k_\mu \phi^\mu = 0, \quad (6)$$

where,

$$k_\mu = (\omega, \mathbf{k}c), \quad (7)$$

is the wave 4-vector.

We can say that the Lorenz condition (5) has eliminated one of the four possible polarization states, leaving three. In the rest of this note, we suppose that the wave vector \mathbf{k} is in the z -direction. Then we can write one basis for the three remaining polarization states ϵ_μ of the 4-potential as,

$$\epsilon_\mu^{(1)} = (0, 1, 0, 0), \quad (8)$$

$$\epsilon_\mu^{(2)} = (0, 0, 1, 0), \quad (9)$$

$$\epsilon_\mu^{(3)} = (kc/\omega, 0, 0, 1). \quad (10)$$

We now show that for waves with $\omega = kc$, as holds in vacuum, the longitudinal polarization state $\epsilon_\mu^{(3)}$ can be eliminated by a **gauge transformation** (while staying within the Lorenz gauge).³

The (gauge) transformation,

$$\phi_\mu \rightarrow \phi_\mu + \partial_\mu \Omega, \quad \phi \rightarrow \phi + \frac{1}{c} \frac{\partial \Omega}{\partial t}, \quad \mathbf{A} \rightarrow \mathbf{A} - \nabla \Omega, \quad (11)$$

does not change the electromagnetic fields,

$$\mathbf{E} = -\nabla \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \quad (12)$$

and the revised potentials (11) still satisfy the Lorenz condition (5) provided that,

$$\partial_\mu \partial^\mu \Omega = 0 = \frac{1}{c^2} \frac{\partial^2 \Omega}{\partial t^2} - \nabla^2 \Omega. \quad (13)$$

For example, we can choose,

$$\Omega = \frac{1}{ik} e^{ik(z-ct)}, \quad \partial_\mu \Omega = -(1, 0, 0, 1) e^{ik(z-ct)}. \quad (14)$$

²For a survey of several gauge conditions, see [3, 4].

³The choice of a gauge condition is not sufficient for the potentials to be unique. For an example of two rather different sets of potentials in the Lorenz gauge for waves inside a rectangular metallic cavity, see sec. 2.2.3 of [5].

Then, for waves with $\omega = kc$, the revised polarization state 3 vanishes,

$$\epsilon_{\mu}^{(3)} = (1, 0, 0, 1) \rightarrow \epsilon_{\mu}^{(3)} + \partial_{\mu}\Omega = (0, 0, 0, 0). \quad (15)$$

This confirms that familiar result that for electromagnetic waves which obey the free-space relation that $\omega = kc$ there is no longitudinal polarization state. Loosely speaking, we can gauge away the longitudinal polarization of waves in free space, but not for those in matter. *The transverse polarizations states (9)-(10) are altered by the gauge transformation (11), but it is usual to redefine the transverse polarization states after the gauge transformation to have their original forms again.*

2.2 Gravitational Waves

In Einstein's theory of gravitation one considers waves as weak perturbations of the metric tensor,

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi_{\mu\nu}, \quad (16)$$

where $\eta_{\mu\nu}$ is the (Euclidean) metric for empty space,

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (17)$$

as was tacitly assumed in sec. 2.1. For isotropic spacetime, we infer that the ‘‘potential’’ tensor $\phi_{\mu\nu}$ is symmetric, with only 10 independent components. We can enforce a Lorenz-like gauge condition on the derivatives of $\phi_{\mu\nu}$ that,

$$\partial_{\mu}\eta^{\mu\lambda}\phi_{\lambda\nu} = \partial^{\mu}\phi_{\mu\nu}0. \quad (18)$$

The four conditions (18) reduce the number of independent components of $\phi_{\mu\nu}$ to six.

We now wish to argue that further use of gauge transformations, within the Lorenz-like gauge, reduce the number of independent components of $\phi_{\mu\nu}$ to two.

However, arguments based on consideration of waves $\phi_{\mu\nu}$ in otherwise empty space miss a noteworthy issue: that weak gravitational waves inside low-density matter can have five independent polarization states. This is clearer in a quantum view in which gravitational waves are associated with spin-2 quanta, which have five independent spin components, in general. Hence, we infer that there exists one more condition on the $\phi_{\mu\nu}$ which holds even for waves inside low-density matter. Here, we simply state this condition to be that $\phi_{\mu\nu}$ is traceless.

We now consider gravitational waves in free space that propagate in the z -direction,

$$\phi_{\mu\nu} = \epsilon_{\mu\nu} e^{ik(z-ct)}, \quad (19)$$

where $\epsilon_{\mu\nu}$ is the (constant) polarization tensor. The Lorenz-like condition (18) tells us that,

$$k^\mu \epsilon_{\mu\nu} = 0, \quad k_\mu = kc(1, 0, 0, 1), \quad \Rightarrow \epsilon_{0\nu} = \epsilon_{3\nu}. \quad (20)$$

The requirement of gauge invariance of the potentials $\phi_{\mu\nu}$ tells us that the transformation,

$$\phi_{\mu\nu} \rightarrow \phi'_{\mu\nu} = \phi_{\mu\nu} + \partial_\mu \Omega_\nu + \partial_\nu \Omega_\mu \quad (21)$$

does not change the physics provided the 4-vector Ω_μ satisfies the free-space wave equation,

$$\partial_\nu \partial^\nu \Omega_\mu = 0 = \frac{1}{c^2} \frac{\partial^2 \Omega_\mu}{\partial t^2} - \nabla^2 \Omega_\mu. \quad (22)$$

Hence, we can consider,

$$\Omega_\mu = \chi_\mu e^{ik(z-ct)}, \quad (23)$$

for any constant 4-vector χ_μ . Applying the gauge transformation (21) to the wave potentials (20), the transformed polarization states are,

$$\epsilon'_{\mu\nu} = \epsilon_{\mu\nu} + k_\mu \chi_\nu + k_\nu \chi_\mu. \quad (24)$$

We choose the four constants χ_μ to eliminate the $\epsilon'_{0\nu}$:

$$\epsilon'_{00} = \epsilon_{00} + 2k_0 \chi_0 = \epsilon_{00} + 2kc\chi_0, \quad \Rightarrow \chi_0 = -\epsilon_{00}/2kc, \quad (25)$$

$$\epsilon'_{01} = \epsilon_{01} + k_0 \chi_1 + k_1 \chi_0 = \epsilon_{01} + kc\chi_1, \quad \Rightarrow \chi_1 = -\epsilon_{01}/kc, \quad (26)$$

$$\epsilon'_{02} = \epsilon_{02} + k_0 \chi_2 + k_2 \chi_0 = \epsilon_{02} + kc\chi_2, \quad \Rightarrow \chi_2 = -\epsilon_{02}/kc, \quad (27)$$

$$\epsilon'_{03} = \epsilon_{03} + k_0 \chi_3 + k_3 \chi_0 = \epsilon_{03} + kc\chi_3 - \epsilon_{00}/2, \quad \Rightarrow \chi_3 = (\epsilon_{00}/2 - \epsilon_{03})/kc. \quad (28)$$

So far,

$$\epsilon'_{00} = \epsilon'_{01} = \epsilon'_{02} = \epsilon'_{03} = 0. \quad (29)$$

The Lorenz-like condition (20), applied to $\epsilon'_{\mu\nu}$, tells us that,

$$\epsilon'_{30} = \epsilon'_{31} = \epsilon'_{32} = \epsilon'_{33} = 0. \quad (30)$$

Since $\epsilon'_{\mu\nu}$ is symmetric, we also have that,

$$\epsilon'_{10} = \epsilon'_{20} = \epsilon'_{13} = \epsilon'_{23} = 0. \quad (31)$$

The remaining nonzero components are ϵ'_{11} , ϵ'_{22} and $\epsilon'_{12} = \epsilon'_{21}$. Since $\epsilon'_{\mu\nu}$ is traceless, $\epsilon'_{22} = -\epsilon'_{11}$, and the polarization tensor $\epsilon'_{\mu\nu}$ has only two independent (transverse) components, ϵ'_{11} and ϵ'_{12} ,

$$\epsilon'_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon'_{11} & \epsilon'_{12} & 0 \\ 0 & \epsilon'_{12} & -\epsilon'_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (32)$$

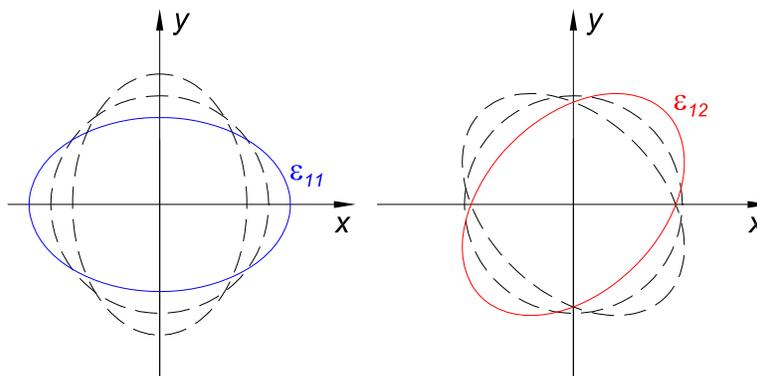
The fact that a massless particle with spin S can have only two spin (polarization) states $S_z = \pm S$ when propagating in the z direction is discussed by Bargmann and Wigner [6] from the perspective of relativistic invariance.

2.2.1 Physical Significance of the Two Polarizations

The gravitational waves parameterized by ϵ_{11} and ϵ_{12} (dropping the 's in eq. (32)) perturb the weak-field metric tensor according to eq. (16), and so affect the invariant interval between two events,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu. \quad (33)$$

If a gravitational plane wave of polarization ϵ_{11} is incident on a massive sphere, the x -separation between pairs of points increases, while the y -separation decreases. As a result, the sphere is (slightly) deformed into an ellipsoid, as sketched on the left below. *Of course, half a wave period later, the x -separation has decreased and the y -separation has increased.*



In contrast, a wave of polarization ϵ_{12} increases the separation between points for which $dx = dy$, and decreases the separation when $dx = -dy$, as shown on the right in the sketch above.

These oscillatory deformations have a quadrupole character, with the two polarization states rotated by 45° with respect to one another (compared to the 90° rotation between x and y linear polarizations of electromagnetic waves). *Likewise, the lowest multipole of gravitational waves emitted by an oscillating mass is the quadrupole.*

References

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