Notes on Pocklington

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Section 1 of the 1897 paper by Pocklington [1] seeks a solution to the wave equation for the electric field $\mathbf{E}$ in free space outside a conductor:

$$\boldsymbol{\nabla}^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}\,, \tag{1}$$

noting also that

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0, \tag{2}$$

and that the electric field is perpendicular to the surface of a good/perfect conductor.

In section 2, Pocklington notes that eq. (1) has solutions that are spherical waves emanating from the origin. In the spirit of Hertz, he states this solution as related to a scalar potential Π that is in some way a generalization of the scalar potential $\phi = 1/r$ that solves Laplace's equation $\boldsymbol{\nabla}^2 \phi = 0$ in electrostatics. The wave version of ϕ is

$$\Pi = \frac{e^{i(kr - \omega t)}}{r}\,, \tag{3}$$

where $k = 2\pi/\lambda$ and $\omega = 2\pi/T = 2\pi c/\lambda$. We should note that

$$\boldsymbol{\nabla}^2 \Pi = \frac{1}{c^2} \frac{\partial^2 \boldsymbol{\Pi}}{\partial t^2} = -k^2 \Pi. \tag{4}$$

Then, the fields

$$E_x = \frac{\partial^2 \Pi}{\partial x \partial z}\,, \qquad E_y = \frac{\partial^2 \Pi}{\partial y \partial z}\,, \qquad E_z = \frac{\partial^2 \Pi}{\partial z^2} + k^2 \Pi, \tag{5}$$

satisfy both eqs. (1) and (2). In particular,

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{E} &= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_x}{\partial y} = \frac{\partial^3 \Pi}{\partial x^2 \partial z} + \frac{\partial^3 \Pi}{\partial y^2 \partial z} + \frac{\partial^3 \Pi}{\partial z^3} + k^2 \frac{\partial \Pi}{\partial z} \\ &= \frac{\partial}{\partial z} \boldsymbol{\nabla}^2 \Pi + k^2 \frac{\partial \Pi}{\partial z} = -k^2 \frac{\partial \Pi}{\partial z} + k^2 \frac{\partial \Pi}{\partial z} = 0 \end{aligned} \tag{6}$$

Pocklington knew from Hertz that the fields (5) can be thought of as due to the a charge near the origin that moves in very small oscillations along the z axis. He now wants to consider the possibility of oscillations along some other direction, say along the $\mathbf{s}$ axis. So, he rewrites eq. (5) as

$$\mathbf{E} = \boldsymbol{\nabla} \frac{\partial \Pi}{\partial s} + k^2 \Pi\ \hat{\mathbf{s}}, \tag{7}$$

which still satisfies both eqs. (1) and (2).

He now supposes that the oscillating charges are distributed along a curving wire, where s is the distance along that wire. He takes function $f(s)$ as a measure of the current in the wire at position s, and integrates eq. (7) along the wire to find

$$\mathbf{E} = \boldsymbol{\nabla} \int ds f(s) \frac{\partial \Pi}{\partial s} + k^2 \int ds f(s) \Pi \, \hat{\mathbf{s}}. \tag{8}$$

The distance r used in the function Π is now the distance to the observer from the point on the wire, rather than the distance from the origin.

He then integrates the first term by parts (and ignores the contributions from the ends of the wire since the current is zero if the wire is open ended = dipole antenna, or the current is the same at both "ends" if the wire forms a loop), yielding

$$\mathbf{E} = -\boldsymbol{\nabla} \int ds \frac{df(s)}{ds} \Pi + k^2 \int ds f(s) \Pi \, \hat{\mathbf{s}}. \tag{9}$$

This note has been expanded upon in [2].

References

[1] H.C Pocklington, *Electrical Oscillations in Wires*, Proc. Camb. Phil. Soc. **9**, 324 (1897), http://kirkmcd.princeton.edu/examples/EM/pocklington_pcps_9_324_97.pdf

[2] K.T. McDonald, *Currents in a Center-Fed Linear Dipole Antenna* (June 27, 2007), http://kirkmcd.princeton.edu/examples/transmitter.pdf