Angular Momentum of the Photon

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When Planck introduced the quantum of action, now called h, in 1899 [28], it was a rather abstract concept. In 1900 [30], Planck associated $h\nu$ with the quantum of energy of a harmonic oscillator of frequency ν,[1] which could emit electromagnetic radiation in case of an oscillating electric charge. In 1905 [36], Einstein added that the electromagnetic radiation of a charged oscillator consisted of "light quanta" (*lichtquanta*) whose energies are integral multiples of $h\nu$. This revived the Newtonian notion that light has particle-like properties. Since particles can have momentum and angular momentum as well as energy, the implication that particles can have quantized energy should extend to quantized momentum and angular momentum as well, but the acceptance of this was slow.

In 1909, Einstein [39] rather indirectly implied that light quanta of frequency ν have momentum $h\nu/c$ (in the equation on p. 824). In 1917 [52], he stated more crisply that a light quantum of energy E has momentum E/c (p. 49), and identified this momentum as $h\nu/c$ on p. 61. However, this relation was not generally accepted until Compton's analysis (1924) [60] of the scattering of x-rays by electrons, in which electrons initially at rest recoiled with nonzero momentum and kinetic energy.

Meanwhile, a tentative quantum model of atoms was made by Nicholson (1912), who suggested on p. 679 of [42] that the angular momentum of atomic electrons is quantized. This notion was further developed by Bohr (1913), who identified the basic quantum of orbital angular momentum as $\hbar = h/2\pi$ on p. 15 of [43].

In electromagnetic decays of an excited atom, the electromagnetic radiation carries away the difference in angular momentum between the initial and final atomic levels, so if angular momentum is conserved, the angular momentum of the electromagnetic field is also quantized, in integer multiples of $\hbar$. And, for circularly polarized radiation, the angular momentum cannot be zero. This insight may have been first published by Schaposchnikow (1914) [44], working together with Busch [45]. It was then discussed by Abraham (1914) [46], Bohr (1918) [53], Rubinowicz (1918) [54], and appeared in Sec. 6.1 of the 1921 edition of Sommerfeld's *Atombau* [58]. Rubinowicz seems to have been the first to note that ordinary (*i.e.*, electric-dipole) atomic transition involve changes in the atomic angular momentum of only 0 (for linearly polarized radiation) or $\pm\hbar$ (for circularly polarized radiation). The implication that circularly polarized light quanta carry angular momentum $\pm\hbar$ was, however, not emphasized in these early works.

It is stated in [90, 99, 100] that Bose's original, English paper on quantum statistics (June 1924, but now lost) included an argument that there are two independent states of light quanta for a given energy and momentum, corresponding to two (not three) states of angular momentum with magnitude $\hbar$. Bose sent this paper to Einstein, who translated it into German for publication [61], but replaced Bose's argument with the classical observation that light waves have two independent polarization states.[2]

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[1]In 1911, Planck [40, 41] argued for the existence of zero-point energy $h\nu/2$ of an oscillator of frequency ν, such that possible energies are $(n + 1/2)\, h\nu$ for non-negative integer n.

[2]Einstein elaborated on Bose's paper in [63, 64].

Meanwhile, studies of the anomalous Zeeman effect suggested that some atoms have energy levels whose angular momentum has the form $(n + 1/2)\hbar$ for integer n (first noted by Lande (1921) [57]). Already in 1920, Compton [55] had speculated that electrons have quantized angular momentum, of value $\hbar$, but only in 1925 did Uhlenbeck and Goudsmit infer from the anomalous Zeeman effect that the electron has intrinsic "spin" of $\hbar/2$ [66, 68].[3,4,5,6]

Einstein's term "light quantum" became replaced by the now-familiar term "photon" following papers by Lewis in 1926 [70, 72], although the physics of these papers was "non-standard".[7]

Once the notion of electron "spin" became accepted, it was natural consider that light quanta also have "spin", as argued, for example, by Beck (1927) [73], and further discussed by Ruark and Urey (1927) [74].

In 1931, Raman discussed other experimental evidence for the angular momentum of light in three little-known papers [78, 79, 80], citing the experiments of Hanle [75] and Bär [76]. The more well known demonstration of the angular momentum of light is that by Beth (1936) [81].

A Appendix: Energy, Momentum and Angular Momentum of Classical Electromagnetic Fields

This Appendix reviews how the "mechanical" concepts of energy, momentum and angular momentum came to be associated with electromagnetic fields as well.

A.1 Field Energy

Already by 1800, potential energy was associated with gravity, electrostatics and magnetostatics, but no location was assigened to these energies. In 1856, Maxwell argued on p. 63 of [6] that the magnetic potential energy $U_m = \int \rho_m \Psi_m \, d\mathrm{Vol}/2$ could be re-expressed in terms of the magnetic fields $\mathbf{B}$ and $\mathbf{H} = \mathbf{B}/\mu$, where ρ_m is the density of hypothetical magnetic charge, Ψ_m is the magnetic scalar potential and μ is the relative magnetic permeability, as $U_m = \int \mathbf{B} \cdot \mathbf{H} \, d\mathrm{Vol}/8\pi$ (in Gaussian units), and that the density of magnetic field energy at some point in space is $\mathbf{B} \cdot \mathbf{H}/8\pi$.

[3]Uhlenbeck and Goudsmit were inspired in part by Pauli's discussion of a fourth quantum number of electrons in atoms [65], in the paper that introduced the "exclusion principle".

[4]The term "spin" was used in the second paper of Uhlenbeck and Goudsmit [68].

[5]For reminiscences by Goudsmit and Uhlenbeck, see [83, 84, 85], which mention that Kronig had the idea of electron spin in early 1925, but was discouraged from publishing it by Pauli. Bichowsky and Urey [69] also claimed to have invented the notion of electron spin.

[6]Indirect evidence for nonclassical electron "spin" was obtained by Barnett [48] already in 1915, when he measured the ratio L/μ of the angular momentum L to the magnetic moment μ of electrons in iron to be 1/2 of the classical prediction. However, in the same year Einstein and de Haas [49] reported experimental agreement with the classical model, and Barnett's (correct) result was largely ignored.

[7]As recounted, for example, in [96, 98], the term "photon" was first used by Troland (1916) [50, 51] in a physiological context, and was employed in a similar context by Joly (1921) [56]. Usage of the term "photon" closer to its present meaning occurred in France in 1925 [62, 67, 71], apparently inspired by Perrin.

The analogous argument for electric field energy was given by Maxwell in Arts. 630-631 of his *Treatise* [10], *i.e.*, $U_e = \int \rho_e \Psi_e \, d\mathrm{Vol}/2 = \int \mathbf{E} \cdot \mathbf{D} \, d\mathrm{Vol}/8\pi$ such that the density of magnetic field energy at some point in space is $\mathbf{D} \cdot \mathbf{E}/8\pi$.[8]

A.2 Field Momentum

Apparently, Kepler considered the pointing of comets' tails away from the Sun as evidence for radiation pressure of light [34].

Pressure is associated with force, which is associated with time rate of change of momentum, so the notion of radiation pressure has an implication that light is associated with momentum. However, the historical development of this association was not swift.

The topic of radiation pressure was long dormant, until reconsidered by Balfour Stewart in 1871 [9] for the case of thermal radiation.

After his unification of electricity, magnetism and light [7], Maxwell argued (sec. 792 of [10]) that the radiation pressure P of light is equal to its energy density u,

$$P = u = \frac{D^2}{4\pi} = \frac{H^2}{4\pi} \tag{1}$$

for an electromagnetic wave with fields $\mathbf{D}$ and $\mathbf{H}$ in vacuum, but he did not explicitly associate this pressure with momentum in the electromagnetic field.[9]

Building on the concept of Faraday's electrotonic state,[10] Maxwell did have a vision of electromagnetic momentum, computed as [8, 10] (see also [88]),

$$\mathbf{P}_{\mathrm{EM}}^{(\mathrm{Maxwell})} = \int \frac{\rho \mathbf{A}^{(C)}}{c} \, d\mathrm{Vol}, \tag{2}$$

where ρ is the electric charge density and $\mathbf{A}^{(C)}$ is the vector potential in the Coulomb gauge (that Maxwell used prior to the explicit recognition of gauge conditions [89]), but the form (2) seems to associate the momentum with charges rather than with fields.

In 1891, J.J. Thomson noted [17] that a sheet of electric displacement $\mathbf{D}$ (parallel to the surface) which moves perpendicular to its surface with velocity $\mathbf{v}$ must be accompanied by a sheet of magnetic field $\mathbf{H} = \mathbf{v}/c \times \mathbf{D}$ according to the free-space Maxwell equation $\boldsymbol{\nabla} \times \mathbf{H} = (1/c) \, \partial \mathbf{D}/\partial t$.[11] Then, the motion of the energy density of these sheets implies there is also a momentum density, eqs. (2) and (6) of [17],

$$\mathbf{p}_{\mathrm{EM}}^{(\mathrm{Thomson})} = \frac{\mathbf{D} \times \mathbf{H}}{4\pi c} \,. \tag{3}$$

[8]In Sec. 82, p. 492 of [8], Maxwell noted that if this argument were applied to gravity, assuming the existence of a vector gravitational field $\mathbf{G}$, then the density of gravitational field energy would be $-G^2/8\pi$. Maxwell considered this to be physically implausible, and inferred that gravity is not describable by a vector field.

[9]Maxwell (and Thomson and Lorentz and most others influenced by the concept of a material aether), regarded the fields $\mathbf{D}$ and $\mathbf{H}$ as more "basic" than $\mathbf{E}$ and $\mathbf{B}$.

[10]Art. 60 of [2], Art. 1661 of [3], Arts. 1729 and 1733 of [4], and Art. 3269 of [5].

[11]Variants of Thomson's argument were given by Heaviside in 1891, sec. 45 of [18], and much later by Feynman in sec. 18-4 of [82], where it was noted that Faraday's law, $\boldsymbol{\nabla} \times \mathbf{E} = -(1/c) \, \partial \mathbf{B}/\partial t$, combined with the Maxwell equation for $\mathbf{H}$ implies that $v = c$ in vacuum, which point seems to have been initially overlooked by Thomson, although noted by him in Sec. 265 of [26].

Also in 1891, Heaviside identified the momentum of the *free ether* in Sec. 26 of [20] as,[12]

$$\mathbf{p}_{\rm EM}^{\rm (Heaviside)} = \frac{\mathbf{D}\times\mathbf{B}}{4\pi c}\,. \tag{4}$$

This was a clarification of his discussion in 1886, eq. (7a) of [15], of a *magnetoelectric force* $\mathbf{D}/4\pi c \times \partial\mathbf{B}/\partial t$.[13]

In 1893, Thomson transcribed much of his 1891 paper into the beginning of *Recent Researches* [23], adding the remark (p. 9) that the momentum density (3) is closely related to the Poynting vector [13, 14],[14,15]

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}. \tag{5}$$

The form (3) was also used by Poincaré in 1900 [29], following Lorentz' convention [21] that the force on electric charge q be written $q(\mathbf{D} + \mathbf{v}/c \times \mathbf{H})$, and that the Poynting vector is $(c/4\pi)\,\mathbf{D}\times\mathbf{H}$. In 1903 Abraham [33] argued for,

$$\mathbf{p}_{\rm EM}^{\rm (Abraham)} = \frac{\mathbf{E}\times\mathbf{H}}{4\pi c} = \frac{\mathbf{S}}{c^2}\,, \tag{6}$$

and in 1908 Minkowski [37] advocated the form,[16,17]

$$\mathbf{p}_{\rm EM}^{\rm (Minkowski)} = \frac{\mathbf{D}\times\mathbf{B}}{4\pi c}\,. \tag{7}$$

Thomson did not relate the momentum density (3) to the radiation pressure of light, eq. (4), until 1904 (p. 355 of [35]) when he noted that $P = F/A = c\,p_{\rm EM} = D^2/4\pi = H^2/4\pi$ for fields moving with speed c in vacuum, for which $D = H$. He also gave an argument (p. 348 of [35]) that the forms (2) and (3) for field momentum are equivalent once the sources of the fields are taken into account.[18]

A.3 Field Angular Momentum

In 1904, J.J. Thomson [35] considered a (Gilbertian) magnetic (mono)pole p and electric charge q, both at rest.

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[12]See also p. 557 of [25] and p. 495 of [19].

[13]Heaviside also mentioned this concept in 1889 on pp. 399-330 of [16].

[14]Thomson argued, in effect, that the field momentum density (3) is related by $\mathbf{p}_{\rm EM} = \mathbf{S}/c^2 = u\mathbf{v}/c^2$ [17, 23]. See also eq. (19), p. 79 of [22], and p. 6 of [47]. It turns out that the energy flow velocity defined by $\mathbf{v} = \mathbf{S}/u$ can exceed c (see, for example, sec. 2.1.4 of [92] and sec. 4.3 of [93].

[15]The idea that an energy flux vector is the product of energy density and energy flow velocity seems to be due to Umov [11], based on Euler's continuity equation [1] for mass flow, $\boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = -\partial\rho/\partial t$.

[16]Minkowski, like Poynting [13], Heaviside [14] and Abraham [33], wrote the Poynting vector as $\mathbf{E}\times\mathbf{H}$. See eq. (75) of [37].

[17]For some remarks on the "perpetual" Abraham-Minkowski debate see [94].

[18]Possibly, Thomson delayed publishing the relation of radiation pressure to his expression (3) until he could demonstrate its equivalence to Maxwell's form (2). For other demonstrations of this equivalence, see [91], and Appendix B of [95].

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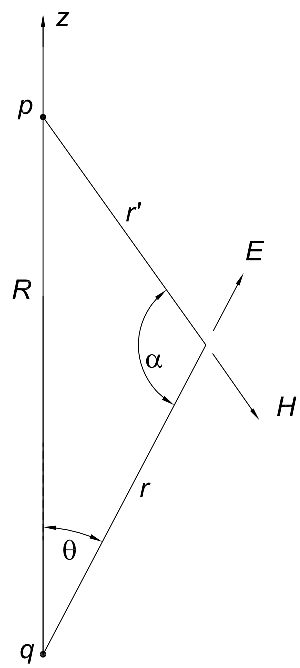

Suppose the electric charge q is at the origin, and the magnetic pole p at distance R away along the positive z-axis, as shown in the figure above. Then, the (Abraham) field-momentum density is, in spherical coordinates $(r, \theta\, \phi)$,

$$\mathbf{p}_{\rm EM} = \frac{\mathbf{E} \times \mathbf{H}}{4\pi c} = \frac{pq \sin\alpha}{4\pi c\, r^2 r'^2}\, \hat{\boldsymbol{\phi}} = \frac{pqR \sin\theta}{4\pi c\, r^2 r'^3}\, \hat{\boldsymbol{\phi}}\,, \tag{8}$$

noting that $H = p/r'^2$ for the magnetic pole, and that $\sin\alpha/R = \sin\theta/r'$ by the sine law.[19]

Thomson also considered the angular momentum in the electromagnetic fields of the pole plus charge,

$$\mathbf{L}_{\rm EM} = \int \mathbf{r} \times \mathbf{p}_{\rm EM}\, d{\rm Vol} = -\frac{pqR}{4\pi c} \int \frac{r \sin\theta}{r^2 r'^3}\, d{\rm Vol}\, \hat{\boldsymbol{\theta}}. \tag{9}$$

This has only a nonzero z-component,

$$\begin{aligned} L_{{\rm EM},z} &= \frac{pqR}{2c} \int_{-1}^{1} \sin^2\theta\, d\cos\theta \int_0^\infty \frac{r\, dr}{(r^2 - 2rR\cos\theta + R^2)^{3/2}} \\ &= \frac{pqR}{2c} \int_{-1}^{1} \sin^2\theta\, d\cos\theta\, \frac{1+\cos\theta}{R\sin^2\theta} = \frac{pq}{c}\,, \end{aligned} \tag{10}$$

using Dwight 380.013. The angular momentum vector $\mathbf{L}_{\rm EM}$ points from the electric charge q to the magnetic pole p.

[19] The electromagnetic momentum (8) circulates azimuthally, such that the total electromagnetic momentum $\mathbf{P}_{\rm EM}$ is zero, $\mathbf{P}_{\rm EM} = \int \mathbf{p}_{\rm EM}\, d{\rm Vol} = 0$. Further, the total electromagnetic-field momentum for any configuration of static magnetic poles and electric charges is zero, being the sum of the momenta of all pairs of such particles. Hence, Thomson demonstrated the notable fact that electromagnetic field momentum can be nonzero only if electric charges, or Gilbertian magnetic poles (should they exist), are in motion. See also [86].

In 1904 the notion of quantizing angular momentum was still years away, and the provocative result (10), that the angular momentum of a magnetic pole plus electric charge is independent of their separation, went unremarked until 1931 when Dirac [77] argued that $pq/c = \hbar/2$. See also sec. 6.12 of [87].[20]

The next discussions of classical electromagnetic-field angular momentum may have been by Poynting (1909) [38] and by Abraham (1914) [46].

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[This meeting featured an inspirational address by W. Thomson (later Lord Kelvin) as

[20]The constant of the motion (10) had been deduced by Darboux in 1878 [12] and by Poincaré in 1896 [27], but this was not recognized as field angular momentum by them [97].

a memorial to Herschel; among many other topics Thomson speculates on the size of atoms, on the origin of life on Earth as due to primitive organisms arriving in meteorites, and on how the Sun's source of energy cannot be an influx of matter as might, however, explain the small advance of the perihelion of Mercury measured by LeVerrier.];
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Stewart argued that the radiation resistance felt by a charge moving through blackbody radiation should vanish as the temperature of the bath went to zero, just as he expected the electrical resistance of a conductor to vanish at zero temperature. [The 43rd meeting was also the occasion of a report by Maxwell on the exponential atmosphere as an example of statistical mechanics (pp. 29-32), by Rayleigh on the diffraction limit to the sharpness of spectral lines (p. 39), and perhaps of greatest significance to the attendees, a note by A.H. Allen on the detection of adulteration of tea (p. 62).]

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