Princeton University

Ph501
Electrodynamics
Problem Set 2

Kirk T. McDonald

(1998)

kirkmcd@princeton.edu

http://physics.princeton.edu/~mcdonald/examples/

1. Show that the electromagnetic energy of a dielectric subject to fields $\mathbf{E}$ and $\mathbf{D} = \epsilon\mathbf{E}$ is

$$U = \frac{1}{8\pi}\int \mathbf{E}\cdot\mathbf{D}\,d\text{Vol}, \tag{1}$$

by considering the model of atoms as springs (Problem 8b, set 1). The energy U then has two parts:

$$U_1 = \frac{1}{8\pi}\int \mathbf{E}^2\,d\text{Vol}, \tag{2}$$

stored in the electric field, and

$$U_2 = \int n\frac{kx^2}{2}\,d\text{Vol}, \tag{3}$$

stored in the spring-like atoms (n is the number of atoms per unit volume). Assume n is small so that the dielectric constant ϵ is nearly 1.

2. (a) Show that the energy of a quadrupole in an external electric field $\mathbf{E}$,

$$U_{\text{quad}} = -\frac{1}{6} Q_{ij} \frac{\partial E_j}{\partial x_i}, \tag{4}$$

in terms of its quadrupole tensor Q_{ij}, can be rewritten as

$$U_{\text{quad}} = -\frac{Q_{xx}}{4} \frac{\partial E_x}{\partial x}, \tag{5}$$

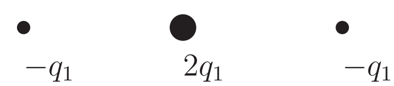
if the quadrupole is rotationally symmetric about the x axis. Give an expression for the force $\mathbf{F}$ on the quadrupole.

(b) A rotationally symmetric quadrupole of strength Q_{xx} (zero net charge, zero dipole moment) is located at distance r from a point charge q. What is the force on the quadrupole if:

i. The x axis is along the line joining Q_{xx} and q?

ii. The x axis is perpendicular to the line joining Q_{xx} and q?

For your own edification, confirm your answer by considering the simple quadrupole:



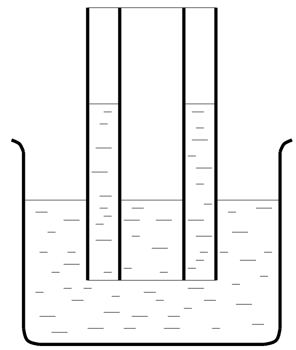
3. The principle of an electrostatic accelerator is that when a charge e escapes from a conducting plane that supports a uniform electric field of strength E_0, then the charge gains energy eE_0d as it moves distance d from the plane. Where does this energy come from?

   Show that the mechanical energy gain of the electron is balanced by the decrease in the electrostatic field energy of the system.

4. (a) Two point dipoles of strength p are aligned along their line of centers, and distance $2d$ apart. Calculate the force between the dipoles via $\mathbf{F} = (\mathbf{p} \cdot \boldsymbol{\nabla})\mathbf{E}$, and by means of the Maxwell stress tensor.

   (b) A spherical conducting shell of radius a carries charge q. It is in a region of zero external field. Calculate the force between two hemispheres in two different ways.

5. (a) Two coaxial pipes of radii a and b ($a < b$) are lowered vertically into an oil bath:



If a voltage V is applied between the pipes, show that the oil rises to height

$$h = \frac{(\epsilon - 1)V^2}{4\pi\rho g \ln\left(\frac{b}{a}\right)(b^2 - a^2)}, \tag{6}$$

where g is the acceleration due to gravity.

(b) Recalling prob. 1(c) of set 1, discuss qualitatively how the force arises the pulls the liquid up into the capacitor.

6. According to a theorem of Green, the potential $\phi(\mathbf{x})$ in the interior of a volume V can be deduced from a knowledge of the charge density $\rho(\mathbf{x})$ inside that volume plus knowledge of the potential and the normal derivative $\partial\phi/\partial n$ of the potential on the surface S that bounds the volume,

$$\phi(\mathbf{x}) = \int_V \frac{\rho(\mathbf{x}')}{R}\, d\mathrm{Vol}' + \frac{1}{4\pi}\int_S \left[\phi(\mathbf{x}')\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) - \frac{1}{R}\frac{\partial\phi(\mathbf{x}')}{\partial n'}\right]\, dS', \tag{7}$$

where $R = |\mathbf{x} - \mathbf{x}'|$ is the distance between the point of observation and the element of the integrand. However, further insights of Green indicate that it suffices to specify only one of ϕ or $\partial\phi/\partial n$ on the bounding surface to determine the potential within. As a particular example, show that the potential within a charge-free sphere of radius a, centered on the origin, can be determined from knowledge of only the potential ϕ on its surface according to (Poisson, 1820)

$$\phi(\mathbf{x}) = \frac{a^2 - x^2}{4\pi a}\int_S \frac{\phi(\mathbf{x}')}{R^3}\, dS'. \tag{8}$$

Green (1828) gave a derivation of Poisson's integral (8) that can be generalized to many other problems in electrostatics. Recall that a key step towards eq. (7) is the identity

$$\begin{aligned}\int_V \boldsymbol{\nabla}\cdot(\psi\boldsymbol{\nabla}\phi - \phi\boldsymbol{\nabla}\psi)\, d\mathrm{Vol} &= \int_V (\psi\nabla^2\phi - \phi\nabla^2\psi)\, d\mathrm{Vol} \\ = \int_S (\psi\boldsymbol{\nabla}\phi - \phi\boldsymbol{\nabla}\psi)\cdot d\mathbf{S} &= \int_S \left(\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right)\, dS.\end{aligned} \tag{9}$$

For problems in which the interior of volume V is charge free the potential obeys $\nabla^2\phi = 0$ there. To have a nonzero potential ϕ inside V there must, of course, be charges on the surface of or exterior to volume V. If function ψ also obeys $\nabla^2\psi = 0$ inside V (and so might be the potential for some other distribution of charges exterior to V), then the identity (9) reduces to

$$0 = \int_S \left(\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right)\, dS. \tag{10}$$

Hence, we could combine eqs. (7) and (10) to yield the relation

$$\begin{aligned}\phi(\mathbf{x}) &= \frac{1}{4\pi}\int_S \left[\phi(\mathbf{x}')\frac{\partial}{\partial n'}\left(\frac{1}{R} + \psi\right) - \left(\frac{1}{R} + \psi\right)\frac{\partial\phi(\mathbf{x}')}{\partial n'}\right]\, dS' \\ &= \frac{1}{4\pi}\int_S \left[\phi(\mathbf{x}')\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'} - G(\mathbf{x},\mathbf{x}')\frac{\partial\phi(\mathbf{x}')}{\partial n'}\right]\, dS',\end{aligned} \tag{11}$$

where

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{R} + \psi. \tag{12}$$

IF the Green's function $G(\mathbf{x}, \mathbf{x}')$ vanishes on the surface S, then we have the desirable relation between the potential ϕ in the interior of V and its value on the bounding surface S,

$$\phi(\mathbf{x}) = \frac{1}{4\pi} \int_S \phi(\mathbf{x}') \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'} \, dS'. \tag{13}$$

Green noted that the auxiliary potential ψ can be thought of as due to exterior charges that bring the surface S to zero potential when there is unit charge at position $\mathbf{x}$ inside volume V, and G as the total potential of that charge configuration. Further, we may think of the bounding surface S as being a grounded conductor for the purposes of determining the potentials ψ and G, in which case the "exterior" charges reside on the surface S. Hence, it is plausible that these exist for interesting physical surfaces S (although it turns out that mathematicians have constructed examples of surfaces for which a Green's function does not exist).

Since the function G is the potential for a specifiable charge configuration, the normal derivative $-\partial G/\partial n$ corresponds to the electric field (whose only nonzero component is E_n) at the surface S produced by those charges. If we consider surface S to be a grounded conductor when determining function G, then the charge density σ_G at position $\mathbf{x}'$ on that surface, caused by the hypothetical unit charge at $\mathbf{x}$, would be $\sigma_G(\mathbf{x}, \mathbf{x}') = E_n/4\pi = -(1/4\pi)\partial G/\partial n$. Green emphasized this phyisical interpretation in his original work, and wrote eq. (13) as

$$\phi(\mathbf{x}) = -\int_S \sigma_G(\mathbf{x}, \mathbf{x}')\phi(\mathbf{x}') \, dS'. \tag{14}$$

Turning at last to Poisson's integral (8), we see that the needed Green's function for a sphere corresponds to the potential at $\mathbf{x}'$ due to unit charge at $\mathbf{x}$ in the presence of a grounded conducting sphere of radius a. Use the method of images to construct the Green's function and its normal derivative, and thereby verify Poisson's result.

7. A parallel-plate capacitor is connected to a battery which maintains the plates at constant potential difference V_0. A slab of dielectric constant ϵ is inserted between the plates, completely filling the space between them.

   (a) Show that the battery does work $Q_0 V_0(\epsilon - 1)$ during the insertion process, if Q_0 is the charge on the plates before the slab is inserted.

   (b) What is the change in the electrostatic energy of the capacitor?

   (c) How much work is done by the mechanical forces on the slab when it is inserted? Is this work done by, or on, the agent inserting the slab?

   Suppose the battery was disconnected before the dielectric was inserted.

   (d) Repeat (b).

   (e) Repeat (c).

8. (a) Find the "escape velocity" of an electron initially 1 A above a grounded conducting plate.

   (b) Point electric dipoles $\mathbf{p}_1$ and $\mathbf{p}_2$ lie in the same plane at a fixed distance apart. If $\mathbf{p}_1$ makes angle θ_1 to their line of centers, show that the equilibrium angle θ_2 of $\mathbf{p}_2$ is related to θ_1 by

$$\tan\theta_1 = -2\tan\theta_2. \tag{15}$$

9. We may define the capacity of a single conductor with respect to infinity as $C = Q/V$, where V is the potential (with respect to potential $\phi = 0$ at ∞) when charge Q is present on the conductor.

   Calculate the capacity of a conductor composed of two tangent spheres of radius a.

10. A grounded conducting sphere of radius a is placed in a uniform external field $\mathbf{E} = E_0\hat{\mathbf{z}}$. (This field changes after the sphere is added.)

    This problem may be solved by the method of images if we suppose the field $\mathbf{E}_0$ is due to two charges $\pm Q$ at positions $z = \pm R$, with Q and R appropriately large.

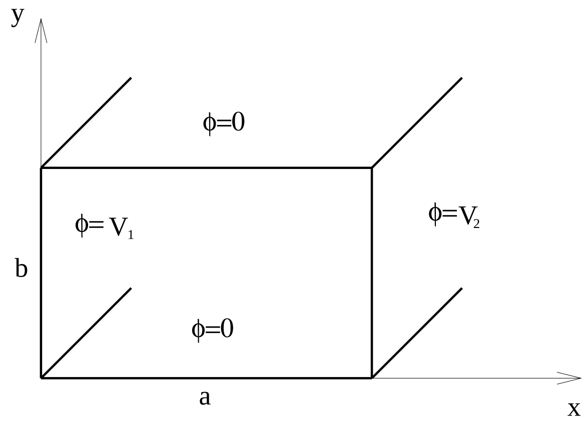
    (a) Show that the image of the source of $\mathbf{E}_0$ is then a dipole $\mathbf{p} = a^3\mathbf{E}_0$ located at the center of the sphere.

    (b) Give an expression for the potential $\phi(r,\theta)$ in spherical coordinates (r,θ,φ) centered on the sphere. Sketch the electric field lines.

    (c) Show that the induced charge distribution on the sphere is $\sigma = \frac{3}{4\pi}E_0\cos\theta$.

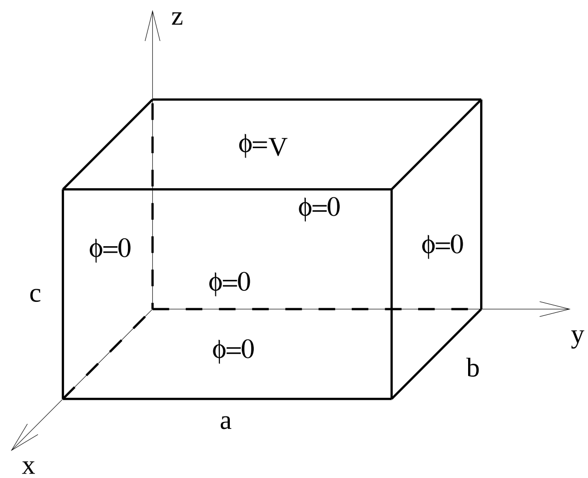
    (d) Show that the force between the two hemispheres with equator perpendicular to $\mathbf{E}_0$ is $F = \frac{9}{16}a^2E_0$.

11. A hollow infinite rectangular conducting tube of sides a and b has two faces grounded and two faces at potentials V_1 and V_2 as shown:



Find the potential $\phi(x, y)$ inside the tube. Remember to use a sum of products of all solutions to the separated equations which do not violate the boundary conditions.

12. A hollow rectangular conducting box has walls at $x = 0$ and a, at $y = 0$ and b, and at $z = 0$ and c. All faces are grounded except that at $z = c$, for which $\phi = V$:



Find the potential $\phi(x, y, z)$ inside the box.

(Choose the signs of the separation constants carefully!)

Solutions

1. The energy stored in a dielectric composed of spring like atoms can be written in two parts,

$$U = U_1 + U_2 = \frac{1}{8\pi}\int \mathbf{E}^2 \; d\text{Vol} + \int n\frac{kx^2}{2} \; d\text{Vol}, \tag{16}$$

where $\mathbf{E}$ is the applied electric field, and where n is the number of molecules per unit volume.

The displacement x in the spring-like atom is related by $kx = eE_{\text{on atom}}$, where e is the charge of an electron. Then,

$$U_2 = \int n\frac{e^2E^2_{\text{on atom}}}{2k} \; d\text{Vol} = \frac{1}{2}\int n\frac{e^2E^2_{\text{on atom}}}{m\omega^2} \; d\text{Vol} = \frac{1}{2}\int n\alpha E^2_{\text{on atom}} \; d\text{Vol}, \tag{17}$$

where $\omega = \sqrt{k/m}$ is the frequency of oscillation of the electron of mass m, and $\alpha = e^2/m\omega^2$ is the atomic polarizability introduced in eq. (69) of set 1.

On p. 20 of the Notes, we argued that $\mathbf{E}_{\text{on atom}} = \mathbf{E} + 4\pi\mathbf{P}/3$, in terms of the applied field $\mathbf{E}$ and the induced polarization $\mathbf{P}$. But, $\mathbf{P} = n\alpha\mathbf{E}_{\text{on atom}}$, so

$$\mathbf{P} = \frac{n\alpha}{1 - 4\pi n\alpha/3}\mathbf{E}, \qquad \text{and} \qquad \mathbf{E}_{\text{on atom}} = \mathbf{E}\left(1 + \frac{n\alpha}{1 - 4\pi n\alpha/3}\right) \approx \mathbf{E}, \tag{18}$$

where the approximation holds for small n. In this case,

$$U_2 \approx \frac{1}{8\pi}\int 4\pi n\alpha E^2 \; d\text{Vol}, \tag{19}$$

and

$$U \approx \frac{1}{8\pi}\int (1 + 4\pi n\alpha)E^2 \; d\text{Vol} \approx \frac{1}{8\pi}\int \epsilon E^2 \; d\text{Vol} = \frac{1}{8\pi}\int \mathbf{E}\cdot\mathbf{D} \; d\text{Vol}, \tag{20}$$

using the Lorenz-Lorentz approximation for the dielectric constant ϵ in terms of the polarizability α, and supposing that $\mathbf{D} = \epsilon\mathbf{E}$.

2. (a) As argued on p. 13 of the Notes, rotational symmetry of a charge distribution about the x axis implies that its quadrupole tenson Q_{ij} can be written

$$Q_{ij} = \begin{pmatrix} Q_{xx} & 0 & 0 \\ 0 & -Q_{xx}/2 & 0 \\ 0 & 0 & -Q_{xx}/2 \end{pmatrix}, \tag{21}$$

and hence, from (4),

$$U = -\frac{Q_{xx}}{6}\left(\frac{\partial E_x}{\partial x} - \frac{1}{2}\frac{\partial E_y}{\partial y} - \frac{1}{2}\frac{\partial E_z}{\partial z}\right) = -\frac{Q_{xx}}{6}\left(\frac{3}{2}\frac{\partial E_x}{\partial x} - \frac{1}{2}\boldsymbol{\nabla}\cdot\mathbf{E}\right) = -\frac{Q_{xx}}{4}\frac{\partial E_x}{\partial x}, \tag{22}$$

using $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$, assuming that the external field is produced by charges not at the location of the quadrupole.

The force on the quadrupole is:

$$\mathbf{F} = -\boldsymbol{\nabla}U = \frac{Q_{xx}}{4}\left(\frac{\partial^2 E_x}{\partial x^2}, \frac{\partial^2 E_x}{\partial x \partial y}, \frac{\partial^2 E_x}{\partial x \partial z}\right). \tag{23}$$

(b) i. Consider a point charge q at at the origin and the quadrupole at $(x, y, z) = (R, 0, 0)$. The x-component of the electric field from q observed at (x, y, z) is

$$E_x = \frac{qx}{r^3}, \qquad \text{where} \qquad r^2 = x^2 + y^2 + z^2. \tag{24}$$

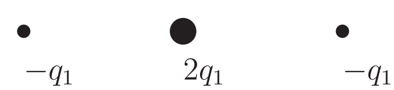
Then,

$$\frac{\partial E_x}{\partial x} = q\frac{r^2 - 3x^2}{r^5}, \tag{25}$$

and the force is evaluated from (23) at $(R, 0, 0)$ as

$$\mathbf{F} = \left(\frac{3}{2}\frac{qQ_{xx}}{R^4}, 0, 0\right). \tag{26}$$

Let us check this for the simple quadrupole shown in the picture.



Suppose the distance between $-q_1$ and $2q_1$ is a. The force on the quadrupole due to charge q at distance R from the center of the quadrupole, and along the latter's axis, is

$$F_x = -\frac{q_1 q}{(R-a)^2} + \frac{2q_1 q}{R^2} - \frac{q_1 q}{(R+a)^2} = -\frac{6a^2 q_1 q}{R^4}(1 + \mathcal{O}(a/R)). \tag{27}$$

This agrees with (26), since

$$Q_{ij} = \int \rho'(3r_i' r_j' - r'^2\delta_{ij})\, d\text{Vol}' \quad \Rightarrow \quad Q_{xx} = \sum 2q' r'^2 = -4q_1 a^2. \tag{28}$$

ii. If, instead, the quadrupole is at $(0, R, 0)$ (but still oriented parallel to the x axis), eqs. (23) and (25) combine to reveal that only the derivative $\partial^2 E_x/\partial x \partial y$ is nonvanishing, and

$$\mathbf{F} = \left(0, -\frac{3qQ_{xx}}{4R^4}, 0\right). \tag{29}$$

Again, we can directly compute the force on the simple quadrupole:

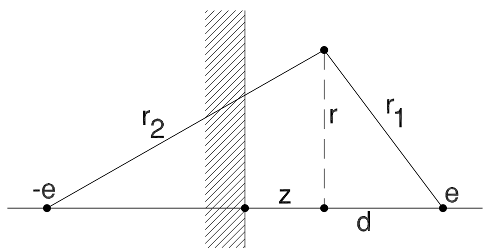
$$F_y = 2\left(\frac{q_1 q}{R^2} - \frac{q_1 q R}{(R^2 + a^2)^{3/2}}\right) \approx \frac{3qq_1 a^2}{R^4} = -\frac{3qQ_{xx}}{4R^4}, \tag{30}$$

using (28).

3. Once the charge has reached distance d from the plane, the static electric field $\mathbf{E}_e$ at an arbitrary point $\mathbf{r}$ due to the charge can be calculated by summing the field of the charge plus its image charge,

$$\mathbf{E}_e(\mathbf{r}, d) = \frac{e\mathbf{r}_1}{r_1^3} - \frac{e\mathbf{r}_2}{r_2^3}, \tag{31}$$

where $\mathbf{r}_1$ ($\mathbf{r}_2$) points from the charge (image) to the observation point $\mathbf{r}$, as illustrated below. The total electric field is then $E_0\hat{\mathbf{z}} + \mathbf{E}_e$.



The charge e and its image charge $-e$ at positions $(r, \theta, z) = (0, 0, \pm d)$ with respect to a conducting plane at $z = 0$. Vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are directed from the charges to the observation point $(r, 0, z)$.

It turns out to be convenient to use a cylindrical coordinate system, where the observation point is $\mathbf{r} = (r, \theta, z) = (r, 0, z)$, and the charge is at $(0, 0, d)$. Then,

$$r_{1,2}^2 = r^2 + (z \mp d)^2. \tag{32}$$

The part of the electrostatic field energy that varies with the position of the charge is the interaction term,

$$\begin{aligned}
U_{\rm int} &= \int \frac{E_0\hat{\mathbf{z}} \cdot \mathbf{E_e}}{4\pi}\, d\mathrm{Vol} \\
&= \frac{eE_0}{4\pi} \int_0^\infty dz \int_0^\infty \pi dr^2 \left(\frac{z-d}{[r^2 + (z-d)^2]^{3/2}} - \frac{z+d}{[r^2 + (z+d)^2]^{3/2}} \right) \\
&= \frac{eE_0}{4} \int_0^\infty dz \left(\left\{ \begin{array}{ll} 2 & \text{if } z > d \\ -2 & \text{if } z < d \end{array} \right\} - 2 \right) \\
&= -eE_0 \int_0^d dz = -eE_0 d.
\end{aligned} \tag{33}$$

When the particle has traversed a potential difference $V = E_0 d$, it has gained energy eV and the electromagnetic field has lost the same energy.

In a practical "electrostatic" accelerator, the particle is freed from an electrode at potential $-V$ and emerges with energy eV in a region of zero potential. However, the particle could not be moved to the negative electrode from a region of zero potential by purely electrostatic forces unless the particle lost energy eV in the process, leading to zero overall energy change. An "electrostatic" accelerator must have an essential component (such as a battery) that provides a nonelectrostatic force that can absorb the energy extracted from the electrostatic field while moving the charge from potential zero, so as to put the charge at rest at potential $-V$ prior to acceleration.

4. (a) First, we calculate the force directly. The electric field from one of the dipoles, taken to be at the origin and with moment $\mathbf{p} = p\hat{\mathbf{x}}$, is

$$\mathbf{E} = \frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{r^3} = \frac{3px\hat{\mathbf{r}}}{r^4} - \frac{p\hat{\mathbf{x}}}{r^3}. \tag{34}$$

For a second dipole at $(x, y, z) = (2d, 0, 0)$, also with moment $\mathbf{p} = p\hat{\mathbf{x}}$, we have

$$\mathbf{F} = (\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{E} = p\left.\frac{\partial\mathbf{E}}{\partial x}\right|_{(2d,0,0)} = -\frac{3p^2}{8d^4}\hat{\mathbf{x}}. \tag{35}$$

The minus sign indicates that the dipole's attract.

As an aside, we can also calculate $\mathbf{F} = -\boldsymbol{\nabla}U$, where U is the energy of interaction of the two dipoles. First, the energy of a charge q_2 at position $\mathbf{r}_2$ in the field of a dipole $\mathbf{p}_1$ at position $\mathbf{r}_1$ is

$$U = q_2\frac{\mathbf{p}_1\cdot\mathbf{r}}{r^3}, \tag{36}$$

where $r = |\mathbf{r}| = |\mathbf{r}_2 - \mathbf{r}_1|$, as on p. 12 of the Notes. A point dipole $\mathbf{p}_2$ is the limit of a pair of charges $\pm q_2$ at positions $\mathbf{r}_2$ and $\mathbf{r}_2 - \mathbf{s}$ where $\mathbf{s} = s\hat{\mathbf{p}}_2$, and the product $q_2 s$ is held constant at value p_2. Thus, the interaction energy of two point dipoles is obtained from (36) as

$$U = \lim_{s\to 0,\ q_2 s = p} q_2\left(\frac{\mathbf{p}_1\cdot\mathbf{r}}{r^3} - \frac{\mathbf{p}_1\cdot\mathbf{r}'}{r'^3}\right) = (\mathbf{p_2}\cdot\boldsymbol{\nabla}_2)\frac{\mathbf{p}_1\cdot\mathbf{r}}{r^3}, \tag{37}$$

where $r' = |\mathbf{r}_2 - \mathbf{s} - \mathbf{r}_1|$. For $\mathbf{p}_1 = \mathbf{p}_2 = p\hat{\mathbf{x}}$ separated by distance $2d$ along $\mathbf{x}$, (37) reduces to

$$U = p^2\left.\frac{\partial}{\partial x}\right|_{x=2d}\frac{1}{x^2}. \tag{38}$$

Then, the force $\mathbf{F} = -\boldsymbol{\nabla}U$ is along $\mathbf{x}$ with magnitude

$$F = -p^2\left.\frac{\partial^2}{\partial x^2}\frac{1}{x^2}\right|_{x=2d} = -\frac{3p^2}{8d^4}, \tag{39}$$

as found in (35).

Now, let us calculate the force via the Maxwell stress tensor. The force on the charges within a (closed) surface S is given by

$$F_i = \oint_S T_{ij}dS_j, \tag{40}$$

as on p. 33 of the Notes, where the Maxwell tensor in empty space is given by

$$T_{ij} = \frac{1}{4\pi}\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right). \tag{41}$$

In the problem with two dipoles, it is convenient to choose the surface as the midplane perpendicular to the line connecting two dipoles (the x axis), closing the surface at infinity around one of the dipoles. On this plane ($x = d$) the only nonzero component of $\mathbf{E}$ is E_x, and this is twice E_x from the dipole at $x = 0$. At radius r from $(d, 0, 0)$ in the symmetry plane, the total field is then

$$E_x(r) = 2p\left(\frac{3d^2}{[r^2+d^2]^{5/2}} - \frac{1}{[r^2+d^2]^{3/2}}\right) = 2p\frac{2d^2 - r^2}{[r^2+d^2]^{5/2}}. \tag{42}$$

The Maxwell stress tensor is thus,

$$T_{ij} = \frac{1}{8\pi}\begin{pmatrix} E_x^2 & 0 & 0 \\ 0 & -E_x^2 & 0 \\ 0 & 0 & -E_x^2 \end{pmatrix}. \tag{43}$$

We take our surface element to be $d\mathbf{S} = (2\pi r dr, 0, 0)$ in cylindrical coordinates, the sign of which implies that the surface S encloses the dipole at $x = 0$. Then, (40) and (43) indicate that only F_x is nonzero, and it is given by

$$\begin{aligned} F_x &= \frac{1}{4}\int_0^\infty r\ drE_x^2 = p^2\int_0^\infty r\ dr\frac{(r^2-2d^2)^2}{(r^2+d^2)^5} \\ &= \frac{p^2}{2}\int_0^\infty dt\frac{(t-2d^2)^2}{(t+d^2)^5} = \frac{p^2}{2}\int_0^\infty dt\frac{[(t+d^2)-3d^2]^2}{(t+d^2)^5} \\ &= \frac{p^2}{2}\int_0^\infty dt\left[\frac{1}{(t+d^2)^3} - \frac{6d^2}{(t+d^2)^4} + \frac{9d^4}{(t+d^2)^5}\right] \\ &= \frac{p^2}{2}\left[\frac{1}{2d^4} - \frac{2}{d^4} + \frac{9}{4d^4}\right] = \frac{3p^2}{8d^4}. \end{aligned} \tag{44}$$

This agrees with (35), noting that since the dipoles attract, the force on the dipole at $x = 0$ is in the $+x$ direction.

(b) The electric field outside the conducting sphere of radius a is $\mathbf{E} = q\hat{\mathbf{r}}/r^2$. The pressure (= force per unit area) on the surface charges is $\mathbf{P} = \sigma\mathbf{E}/2$, where σ is the surface charge density; hence, $\mathbf{P} = q^2\hat{\mathbf{r}}/8\pi a^4$. (The coefficient 1/2 is needed because $\mathbf{E}$ is the field outside the surface, while the field inside the sphere is zero, thus the average field inside the charge layer is $\mathbf{E}/2$.) To find the force between two hemispheres, we integrate the component of pressure normal to the equatorial plane ($P\cos\theta$) over one hemisphere:

$$F = \int_0^1 2\pi a^2\ d\cos\theta\frac{q^2\cos\theta}{8\pi a^4} = \frac{q^2}{8a^2}. \tag{45}$$

Now, let us calculate the force using the Maxwell stress tensor. We integrate F_z over the x-y plane separating our sphere into two hemispheres. Since $d\mathbf{S} = (0, 0, 2\pi r\ dr)$ there, and the only nonzero components of $\mathbf{E}$ on that surface are E_x and E_y, only $T_{zz} = -E^2/8\pi = -q^2/8\pi r^4$ contributes to the force. Integrating from $r = a$ to ∞, we find

$$F_z = \int_a^\infty 2\pi r\ drT_{zz} = \frac{q^2}{4}\int_a^\infty \frac{dr}{r^3} = \frac{q^2}{8a^2}, \tag{46}$$

in agreement with (45).

5. (a) The electrical force F required to pull oil of density ρ into a cylindrical capacitor of inner and outer radii a and b, respectively, to height h above the bath is equal to the force of gravity:

$$F = \rho g h(b^2 - a^2). \tag{47}$$

A second relation for F can be computed from the balance of electrical energy, noting that the capacitor is held at constant voltage by a battery. Suppose we increase the height of the oil by δh. Then, work $F\delta h$ is done on the oil, the energy $U = CV^2/2$ stored in the capacitor changes by δU, and the battery loses energy $V\delta Q$. Conservation of energy implies

$$0 = F\delta h + \delta U - V\delta Q. \tag{48}$$

Since $V = Q/C$, we find for constant voltage,

$$V\delta Q = V^2\delta C = 2\delta\left(\frac{CV^2}{2}\right) = 2\delta U. \tag{49}$$

Together, (48) and (49) imply that

$$F = +\left.\frac{\partial U}{\partial h}\right|_V . \tag{50}$$

As the liquid is drawn *into* the capacitor, the energy for this must come from elsewhere; yet, the energy of the capacitor *increases* because the battery loses energy in twice the amount of work done on the liquid.

We now calculate the stored energy U by integrating the electric field energy density. By cylindrical symmetry and Gauss's law, the electric field between the pipes has form $E_r(r) = \alpha/r$, where α is fixed by

$$V = \int_a^b E_r\, dr = \alpha \ln\frac{b}{a}, \qquad \text{or} \qquad \alpha = \frac{V}{\ln\frac{b}{a}}. \tag{51}$$

Suppose the total height of the capacitor (above the bath) is H. Then, the energy of the electric field in the capacitor is:

$$U = \frac{1}{8\pi}\epsilon h \int_a^b E^2 2\pi r\, dr + \frac{1}{8\pi}(H - h)\int_a^b E^2 2\pi r\, dr, \tag{52}$$

where the first term on the right is the contribution from the space filled with the oil whose dielectric constant is ϵ, while the second term is from the empty space above. Evaluating the integrals:

$$2\pi \int_a^b E^2 r\, dr = 2\pi \frac{V^2}{\ln^2\frac{b}{a}} \int_a^b \frac{dr}{r} = 2\pi \frac{V^2}{\ln\frac{b}{a}}, \tag{53}$$

we then find:

$$U = \frac{1}{4}\frac{V^2}{\ln\frac{b}{a}}\left[(\epsilon - 1)h + H\right]. \tag{54}$$

The force is obtained from (50) and (54):

$$F_{el} = \left.\frac{\partial U}{\partial h}\right|_V = \frac{V^2(\epsilon - 1)}{4 \ln \frac{b}{a}}. \tag{55}$$

Equating this to the force of gravity, (47) we obtain the height h of the oil column:

$$h = \frac{(\epsilon - 1)V^2}{4\pi\rho g \ln\left(\frac{b}{a}\right)(b^2 - a^2)}. \tag{56}$$

(b) The force on the liquid arises from the effect of gradients of the electric field on the molecular dipoles in the liquid. The spatially varying electric field $\mathbf{E}$ results in a bulk dielectric polarization given by

$$\mathbf{P} = \chi\mathbf{E} = \frac{\epsilon - 1}{4\pi}\mathbf{E}, \tag{57}$$

where χ is the dielectric susceptibility and ϵ is the dielectric constant. The energy density associated with the induced polarization is

$$u = -\mathbf{P} \cdot \mathbf{E} = -\frac{\epsilon - 1}{4\pi}E^2, \tag{58}$$

and so the force density on the liquid is given by

$$\mathbf{f} = -\boldsymbol{\nabla} u = \frac{\epsilon - 1}{4\pi}\boldsymbol{\nabla} E^2. \tag{59}$$

The gradient $\boldsymbol{\nabla} E^2$ in the fringe field of the capacitor points from the outside to the interior of the capacitor, with a generally vertical component for the liquid below the capacitor in the present problem.

It is interesting to consider a variant on this problem: a capacitor with horizontal plates completely immersed in a dielectric liquid. Here, the fringe fields of the capacitor pull the liquid in from all sides, "trapping" it inside the capacitor. That is, work would be required to pull the liquid out of the capacitor in any direction.

Is this an example of electrostatic trapping – which is claimed not to exist? No! The "trapping" in the direction perpendicular to the capacitor plates is not provided by purely electrostratic fields, but by the material of the capacitor plates (whose stability is not a result of purely electrostatic effects). See prob. 7 of set 4 for further discussion.

We have concluded that the liquid is drawn into the interior of the capacitor and that the liquid near the middle of the capacitor is forced up against the capacitor plates by electrostatic forces on the induced dipoles. If we drill a hole in the center of one capacitor plate, would liquid squirt out? (If yes, we would have a perpetual motion machine.) No, the fringe fields around the hole will pull liquid into the interior of the capacitor creating a static equilibrium much as before.

6. We work from eq. (13), for which we first need the potential ϕ at a point $\mathbf{r}$ inside a grounded conducting sphere of radius a when unit charge is located at $\mathbf{x}$, also inside the sphere. Then we need the normal derivative of this potential on the inner surface of the sphere, *i.e.*when $|\mathbf{r}| = r = a$.

   The image method for a grounded conducting sphere tells us that the potential inside the sphere can be calculated as that due to unit charge at $\mathbf{x}$ together with charge $-a/x$ at position $\mathbf{x}' = a^2\mathbf{x}/x^2$. We denote the angle between vectors $\mathbf{r}$ and $\mathbf{x}$ as θ, so that

$$R = |\mathbf{r} - \mathbf{x}| = \sqrt{r^2 + 2rx\cos\theta + x^2}, \tag{60}$$

   and

$$R' = |\mathbf{r} - \mathbf{x}'| = \sqrt{r^2 + 2r\frac{a^2}{x}\cos\theta + \frac{a^4}{x^2}}. \tag{61}$$

   We see that when $r = a$, then

$$R' = \frac{a}{x}R. \tag{62}$$

   The potential inside the sphere can now be written

$$\phi(\mathbf{r}) = \frac{1}{R} - \frac{a}{R'x}\,, \tag{63}$$

   The normal derivative of the potential on the inner surface of the sphere is the negative of its radial derivative when $r = a$,

$$\frac{\partial\phi}{\partial n} = -\frac{\partial\phi(r=a)}{\partial r} = \frac{a + x\cos\theta}{R^3} - \frac{a[a + (a^2/x)\cos\theta]}{R'^3x} = \frac{a^2 - x^2}{aR^3}\,, \tag{64}$$

   using eq. (62). Inserting this in eq. (13), we obtain Poisson's integral,

$$\phi(\mathbf{x}) = \frac{a^2 - x^2}{4\pi a}\int_S \frac{\phi(\mathbf{x}')}{R^3}\,dS'. \tag{65}$$

7. (a) The capacitance C of a parallel-plate capacitor of area A, gap thickness d and dielectric constant ϵ is

$$C = \frac{\epsilon A}{d} \equiv \frac{Q}{V}. \tag{66}$$

Adding the dielectric increased the capacitance to

$$C_f = \epsilon C_0, \tag{67}$$

and hence the charge also increase, if the voltage is kept fixed. Thus, the work done by the battery as the dielectric is inserted,

$$\Delta W_{batt} = V_0 \Delta Q = V_0^2 \Delta C = V_0^2(\epsilon - 1)C_0 = Q_0 V_0(\epsilon - 1), \tag{68}$$

is positive.

(b) As the dielectric is inserted, the field energy $U = CV^2/2$ stored in the capacitor changes by

$$\Delta U = \frac{1}{2}\Delta C V_0^2 = \frac{1}{2}C_0 V_0^2(\epsilon - 1) = \frac{1}{2}Q_0 V_0(\epsilon - 1). \tag{69}$$

(c) The work done by the battery, (68), is only partly accounted for in increase in the field energy, (69). The rest of the work done by the battery is done *on* the external agent that held the dielectric during insertion (the external agent gained energy):

$$\Delta W_{\text{on agent}} = \frac{1}{2}Q_0 V_0(\epsilon - 1). \tag{70}$$

(d) If the battery had been disconnected before the dielectric was inserted, then the charge Q_0 would be constant. From (66) we see that the final voltage would be only V_0/ϵ. Recalling (67), the change in the electrostatic field energy would then be

$$\Delta U = \frac{1}{2}\epsilon C_0 \left(\frac{V_0}{\epsilon}\right)^2 - \frac{1}{2}C_0 V_0^2 = \frac{1}{2}Q_0 V_0 \left(\frac{1}{\epsilon} - 1\right) < 0. \tag{71}$$

(e) By conservation of energy, the work done on the external agent that held the dielectric during insertion is equal and opposite to the change in stored energy. Hence the work done on the agent is again positive, but now with the value

$$\Delta W_{\text{on agent}} = \frac{1}{2}Q_0 V_0 \frac{\epsilon - 1}{\epsilon}. \tag{72}$$

That is, the dielectric is pulled into the capacitor whether or not the battery is still connected.

8. (a) The field energy associated with an electron at distance r from a grounded conducting plane is 1/2 that associated with the corresponding image charge, *i.e.*, with that electron plus a positron at distance $-r$, in the absence of the conducting plane. Hence,

$$U = -\frac{1}{2}\frac{e^2}{2r} = -\frac{e^2}{4r}. \tag{73}$$

The fields in the image solution have reality only outside the conducting plane; there is no energy associated with the "fictitious" image fields inside the conductor.

Equation (73) indicates that an electron is "bound" to the conducting plane, and so to escape, must have a minimum velocity related by

$$v_{\min} = \sqrt{\frac{2|U|}{m}} = \sqrt{\frac{e^2}{2mr}} = \sqrt{\frac{e^2c^2}{2mc^2r}} = c\sqrt{\frac{r_e}{2r}}, \tag{74}$$

where $r_e = e^2/mc^2 = 2.8 \times 10^{-13}$ cm is the classical electron radius. Thus, for $r = 1$

A,

$$\frac{v_{\min}}{c} = \sqrt{\frac{2.8 \times 10^{-13}}{2 \times 10^{-8}}} = 0.0037. \tag{75}$$

(Notice that the nonrelativistic approximation suffices.)

The "binding energy" can be estimated from (73) as

$$U = -\frac{e^2}{4mc^2r}mc^2 = -\frac{r_e}{r}\frac{mc^2}{4} = -\frac{2.8 \times 10^{-13}}{10^{-8}}\frac{5.11 \times 10^5 \text{ eV}}{4} = -3.6 \text{ eV}. \tag{76}$$

(b) In equilibrium, the torque on dipole $\mathbf{p}_2$ must vanish, and so $\mathbf{p}_2$ will be directed along the electric field created by dipole $\mathbf{p}_1$. The electric field of the latter is given by

$$\mathbf{E} = \frac{3(\mathbf{p}_1 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}_1}{r^3}. \tag{77}$$

The projection of $\mathbf{E}$ onto the line connecting two dipoles is

$$E_\parallel = \mathbf{E} \cdot \hat{\mathbf{r}} = 2\frac{p_1}{r^3}\cos\theta_1. \tag{78}$$

The orthogonal projection is

$$\mathbf{E}_\perp = \mathbf{E} - \mathbf{E}_\parallel, \tag{79}$$

leading to

$$E_\perp = -\frac{p_1}{r^3}\sin\theta_1, \tag{80}$$

where the minus sign indicates that $\mathbf{E}_\perp$ is directed opposite to $\mathbf{p}_{1,\perp}$.

The angle of the field line, and hence of $\mathbf{p}_2$ is

$$\tan\theta_2 = \frac{E_\perp}{E_\parallel} = -\frac{1}{2}\tan\theta_1. \tag{81}$$

9. We solve the problem of the capacity of two tangent, conducting spheres of radii a by the method of images.

   We first find the image-charge distribution needed to bring one sphere to potential V, but leaving the other at zero potential. Then, we complete the solution by superposing the mirror distribution, obtained by reflection symmetry about the plane through the point of tangency of the two spheres.

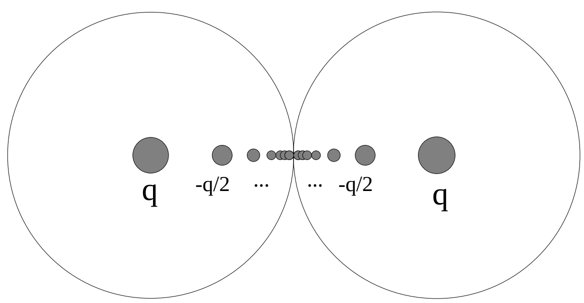
   (a) Place charge q = aV at the center of sphere 1, bringing its surface to otential V.

   (b) To bring sphere 2 to zero potential, place charge $-q(a/2a) = -q/2$ at distance $a^2/2a = a/2$ from the center of sphere 2, following the prescription on p. 41 of the Notes.

   (c) The image charge (b) takes sphere 1 away from potential V. To bring it back, add an image charge (c) inside sphere 1 so that this sphere is at zero potential under the effect of charges (b) and (c). That is, add charge $-(-q/2)(a/(3a/2)) = +q/3$ at distance $a^2/(3a/2) = 2a/3$ from the center of sphere 1, *i.e.*, $a/3$ from the point of contact.

   (d) Add charge $-q/4$ at $3a/4$ from the center of sphere 2 to bring it back to zero potential.

   (e) ....



The total charge needed to bring both spheres to potential V is double that described in the sequence above. Hence,

$$Q = 2q\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...\right) = 2aV \ln 2, \tag{82}$$

and the capacitance is

$$C = Q/V = 2a \ln 2 = 1.386a. \tag{83}$$

Note that since the dimensions of potential are [charge]/[length], capacitance has the dimension of [length] in Gaussian units. Thus, we expect that $C \approx a$ for this problem, since a is the only relevant length.

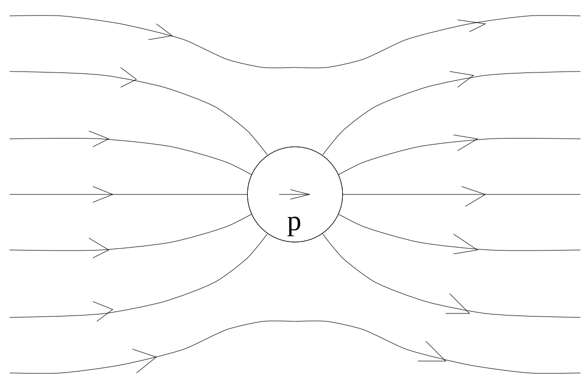
10. (a) The uniform field $\mathbf{E}_0 = E_0\hat{\mathbf{z}}$ is approximated as being due to charges $\pm Q$ at $z = \mp R$, where $Q \to \infty$ and $R \to \infty$ in such a way as to keep Q/R^2 constant. In the limit, the field in the region of the sphere is homogeneous and equal to $\mathbf{E} = (2Q/R^2)\hat{\mathbf{z}}$. According to the image method, we can make the potential on the sphere vanish by adding charge $q' = -Qa/R$ at $z = -a^2/R$ and $-q'$ at $z = a^2/R$. Thus, the perturbation to the field due to the sphere is effectively that due to a dipole with the moment

$$\mathbf{p} = 2\frac{a^2}{R}Q\frac{a}{R}\hat{\mathbf{z}} = a^3\mathbf{E}_0. \tag{84}$$

(b) The potential outside the sphere is thus,

$$\phi = \phi_0 + \phi_{\text{dipole}} = -E_0 r\cos\theta + \frac{E_0 a^3\cos\theta}{r^2}. \tag{85}$$

The field lines bend in to be normal to the sphere at $r = a$:



(c) We find the surface charge density σ from the normal component of the electric field at the surface of the sphere:

$$E_r(a,\theta) = -\left.\frac{\partial\phi}{\partial r}\right|_{r=a} = 3E_0\cos\theta, \tag{86}$$

and so

$$\sigma(\theta) = \frac{E_r(a,\theta)}{4\pi} = \frac{3E_0\cos\theta}{4\pi}. \tag{87}$$

(d) The force acting on the surface charge density σ is $\mathbf{F} = \sigma E_r(a)\hat{\mathbf{r}}/2 = E_r^2(a)\hat{\mathbf{r}}/8\pi$ (where the latter form follows immediately from the Maxwell stress tensor). The force on the right hemisphere is directed along $\mathbf{z}$ and is obtained by integrating the z component of $\mathbf{F}$:

$$F_z = \frac{1}{8\pi}\int_0^1 2\pi a^2\ d\cos\theta(3E_0\cos\theta)^2(\cos\theta) = \frac{9}{16}a^2E_0^2. \tag{88}$$

Since the force on the hemisphere at $z > 0$ is positive, the hemispheres repel each other.

11. We seek solutions to Laplace's equation in 2 dimensions, $\nabla^2\phi(x,y) = 0$, of the form $\phi = X(x)Y(y)$. This leads to solutions of the form $e^{\pm kx}e^{\pm iky}$ or $e^{\pm ikx}e^{\pm ky}$.

    Since the boundary conditions include $\phi = 0$ at $y = 0$ and b, it is advantageous to consider functions Y of the type $e^{\pm iky}$, which can be immediately restricted to the form:

$$Y(y) = \sin ky, \qquad \text{where} \qquad k = n\pi/b, \quad n = 1, 2, \ldots. \tag{89}$$

    This also fixes the separation constants k.

    The general expression for the potential is now:

$$\phi(x,y) = \sum_n X_n(x)Y_n(y) = \sum_n \left(A_n e^{n\pi x/b} + B_n e^{-n\pi x/b}\right) \sin\frac{n\pi y}{b}. \tag{90}$$

    The boundary conditions at $x = 0$ and $x = a$ are

$$\phi(0,y) = V_1 = \sum_n (A_n + B_n) \sin\frac{n\pi y}{b}, \tag{91}$$

$$\phi(a,y) = V_2 = \sum_n \left(A_n e^{n\pi a/b} + B_n e^{-n\pi a/b}\right) \sin\frac{n\pi y}{b}. \tag{92}$$

    A straigthforward approach to find A_n and B_n is to multiply (91) and (92) by $\sin(n\pi y/b)$ and integrate from $y = 0$ to b:

$$\int_0^b \phi(0,y) \sin\frac{n\pi y}{b}\, dy = -\frac{bV_1}{n\pi} \cos\frac{n\pi y}{b}\bigg|_0^b = \frac{bV_1}{n\pi} \begin{cases} 2, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} = \frac{b}{2}(A_n + B_n), \tag{93}$$

    and similarly,

$$\frac{bV_2}{n\pi} \begin{cases} 2, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} = \frac{b}{2}\left(A_n e^{n\pi a/b} + B_n e^{-n\pi a/b}\right). \tag{94}$$

    Thus, for n even, $A_n = B_n = 0$, while for n odd,

$$A_n = \frac{2}{n\pi \sinh\frac{n\pi a}{b}}\left(V_2 - V_1 e^{-n\pi a/b}\right), \qquad B_n = \frac{2}{n\pi \sinh\frac{n\pi a}{b}}\left(V_1 e^{n\pi a/b} - V_2\right). \tag{95}$$

    Finally, we get for the potential:

$$\phi(x,y) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin\frac{n\pi y}{b}}{n\pi \sinh\frac{n\pi a}{b}} \left[V_2 \sinh\frac{n\pi x}{b} + V_1 \sinh\frac{n\pi(a-x)}{b}\right]. \tag{96}$$

    To verify that this solution satisfies the boundary conditions, note that (91) and (93) combine to yield the expansion:

$$1 = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin\frac{n\pi y}{b}. \tag{97}$$

We also note that the potential is symmetric about the midplanes, $\phi(x, y) = \phi(a - x, y) = \phi(x, b - y)$, which could have been invoked as far back as (90) to show that only odd n contributes.

Remark: This problem could also usefully be solved as the superposition of two cases, each with three walls at potential zero and the fourth at a nonzero value. The form of the solution (96) displays this superposition.

12. Since $\phi = 0$ at $x = 0, a$ and $y = 0, b$, solutions $\phi = X(x)Y(y)Z(z)$ must have the form

$$X_m(x) = \sin\frac{m\pi x}{a}, \qquad \text{and} \qquad Y_n(y) = \sin\frac{n\pi y}{b}, \tag{98}$$

where n and m are positive integers (and odd, recalling the remark at the end of problem 9). The functions $Z(z)$ then have the form $e^{\pm kz}$. Since $\phi = 0$ at $z = 0$, we can make the further restriction:

$$Z_{mn}(z) = \sinh k_{mn} z, \tag{99}$$

where k_{mn} is determined by inserting the trial solutions into Laplace's equation, yielding

$$k_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2. \tag{100}$$

The general solution satisfying all the boundary conditions except for the one at the face $z = c$ is:

$$\phi(x, y, z) = \sum_{m,n} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sinh\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\, z. \tag{101}$$

The remaining boundary condition tells us that

$$V = \sum_{m,n} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sinh\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\, c. \tag{102}$$

To find A_{mn}, multiply (102) by $\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$ and integrate from 0 to a in x and from 0 to b in y. Similarly to (93), we find

$$A_{mn} = \frac{16V}{mn\pi^2 \sinh\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\, c}, \tag{103}$$

for odd m and n, and 0 otherwise. Hence,

$$\phi(x, y, z) = \frac{16V}{\pi^2} \sum_{m,n \text{ odd}} \frac{1}{m} \sin\frac{m\pi x}{a} \frac{1}{n} \sin\frac{n\pi y}{b} \frac{\sinh\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\, z}{\sinh\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\, c}. \tag{104}$$

Note that we have demonstrated the expansion

$$1 = \sum_m \frac{4}{m\pi} \sin\frac{m\pi x}{a} \sum_n \frac{4}{n\pi} \sin\frac{n\pi y}{b} \qquad \text{for} \qquad 0 < x < a, \quad 0 < y < b. \tag{105}$$

Since this follows from (97), we could have used it to go from (102) to (103) without performing the integrations.