Ph 206 Problem Set 11

Due: Tuesday April 24, 1984 ; Maximum recorded score = 70 points

① Čerenkov Radiation by a Neutron

A neutron has no charge but it does have magnetic moment $\bar{M}$. Hence we can expect an accelerated neutron to emit radiation. Here we ask whether a neutron travelling with uniform velocity $v > c/n$ inside a dielectric medium will emit Čerenkov radiation?

Towards answering this, we examine our expression for the energy spectrum of the radiation from a pulse:

$$\frac{dU_\omega}{d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \hat{n}\times(\hat{n}\times\bar{\beta})\, e^{i(\omega t' - \bar{k}\cdot\bar{r}')}\,dt'\right|^2$$

A magnetic moment may be thought of as due to a current loop, so we would like a version of the above for a general current rather than just a single point charge. We plausibly identify

$e\bar{\beta} \Leftrightarrow \frac{\bar{J}}{c}\,d\mathrm{vol}$ as the needed relation.

Thus $$\frac{dU_\omega}{d\Omega} = \frac{\omega^2}{4\pi^2 c^3}\left|\iiint \hat{n}\times(\hat{n}\times\bar{J})\, e^{i(\omega t' - \bar{k}\cdot\bar{r}')}\,dt'\,d\mathrm{vol}'\right|^2$$

[In fact, this expression follows directly from pp 181, 182 of Lecture 15, if we note that $|\hat{n}\times(\hat{n}\times\bar{J})| = |\bar{J}\times\hat{n}|$]

For the neutron, we relate the current to the magnetic moment by $\bar{J} = c\bar{\nabla}\times\bar{m}$ (Lecture 8) with $\bar{m}(\bar{r}', t') = \bar{M}\,\delta(x)\delta(y)\delta(z - vt)$ say.

[Strictly speaking, $\bar{J} = c\bar{\nabla}\times\bar{m}$ holds for moving moments only if $\bar{m}$ is $\parallel$ to $\bar{v}$. See Becker sec. 87 for the complications of the general case.]

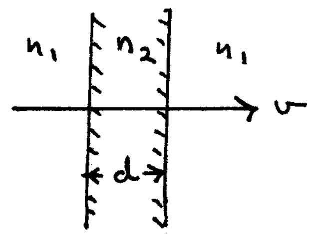
Evaluate $\frac{dU_\omega}{d\Omega}$ in a medium of index n

To show $$\frac{dU_\omega/d\Omega\,|_{\text{moving moment}}}{dU_\omega/d\Omega\,|_{\text{moving charge}}} = \frac{k^2 M^2}{v^2 e^2}$$ k = wave # , M = moment

Hint: integrate by parts to absorb the $\bar{\nabla}$

Since $M_{\text{neutron}} \sim \frac{e\hbar}{\mathcal{M} c}$, $\mathcal{M}$ = mass, we see that the effect is extremely weak!

(2) TRANSITION RADIATION FROM A DIELECTRIC LAYER $\Rightarrow$ 2 SURFACES



FOLLOW THE METHOD OF THE NOTES TO SHOW THAT

$$\frac{dU_\omega}{d\Omega} \sim \frac{4e^2}{\pi^2 c}\left\{\frac{\theta \sin\frac{\omega d}{2c}(\mathrm{I})}{(\mathrm{I})} - \frac{\theta \sin\frac{\omega d}{2c}(\mathrm{II})}{(\mathrm{II})}\right\}^2$$

WHERE $(\mathrm{I}) = \frac{1}{\gamma^2} + \frac{\omega_{p1}^2}{\omega^2} + \theta^2$ ETC.

IN THE LITERATURE I FIND THIS RESULT WRITTEN AS

$$\frac{dU_\omega}{d\Omega} = 4\sin^2\frac{\omega d}{2c}(\mathrm{II}) \cdot \left.\frac{dU_\omega}{d\Omega}\right|_{\text{SINGLE SURFACE}}$$

WHICH SEEMS ONLY AN APPROXIMATION TO OUR RESULT.

(3) BREMSSTRAHLUNG REVISITED.

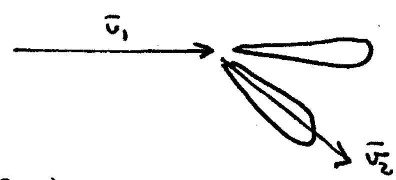
SUPPOSE A CHARGE e HAS INITIAL VELOCITY $\bar{v}_1$, BUT AT $t=0$ IT SUFFERS A BRIEF ACCELERATION WHICH LEAVES IT WITH FINAL VELOCITY $\bar{v}_2$.

SHOW THAT THE SPECTRUM OF RADIATION EMITTED IS

$$\frac{dU_\omega}{d\Omega} = \frac{e^2}{4\pi c^3}\left[\frac{\hat{n}\times(\hat{n}\times\bar{v}_1)}{1-\frac{\bar{v}_1\cdot\hat{n}}{c}} - \frac{\hat{n}\times(\hat{n}\times\bar{v}_2)}{1-\frac{\bar{v}_2\cdot\hat{n}}{c}}\right]^2$$

AT LEAST FOR FREQUENCIES $\omega \ll \frac{1}{\Delta t}$ WHERE Δt = DURATION OF THE ACCELERATION. THIS IS INDEPENDENT OF ω, AS IN LECTURE 20.

AS $v \to c$ WE SEE THAT A SIDEWAYS DEFLECTION LEADS TO 2 PEAKS OF RADIATION, ALONG THE INITIAL AND FINAL DIRECTIONS OF MOTION



DIVIDING THE ABOVE EXPRESSION BY $\hbar$, WE OBTAIN THE NUMBER OF PHOTONS EMITTED (PER FREQUENCY INTERVAL...)

AGAIN $e^2/\hbar c = 1/137$. THIS RESULT IS ESSENTIALLY UNCHANGED IN QUANTUM ELECTRODYNAMICS

④ NEUTRON DECAY A FREE NEUTRON CAN DECAY TO A PROTON + ELECTRON + 'SOMETHING ELSE'.

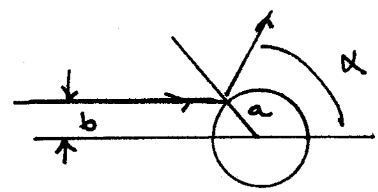
IN OUR CLASSICAL THEORY, THE 'SOMETHING ELSE' MIGHT BE A KIND OF BREMSSTRAHLUNG RADIATION. THE PROTON IS ESSENTIALLY AT REST (FOR THE INITIAL NEUTRON AT REST), BUT THE ELECTRON IS 'EJECTED' (OR 'CREATED') WITH VELOCITY v LARGE). USE THE RESULT OF PROBLEM ③ WITH $v_1 = 0$, $v_2 = v$, AND INTEGRATE OVER $d\Omega$ TO SHOW

$$U_\omega = \frac{e^2}{\pi c}\left(\frac{c}{v}\ln\left(\frac{c+v}{c-v}\right) - 2\right) = \text{CONSTANT INDEPENDENT OF } \omega$$

$$U = \int_0^{\omega_{MAX}} U_\omega \, d\omega = \omega_{MAX} U_\omega$$ IS CERTAINLY LIMITED TO THE TOTAL AVAILABLE ENERGY. HOWEVER, EXPERIMENTALLY THIS RADIATION TAKES ONLY A SMALL FRACTION $\left(\sim \frac{e^2}{\hbar c} \sim \frac{1}{137}\right)$ OF THE AVAILABLE ENERGY – AND IS CALLED A RADIATIVE CORRECTION. AS AN EXPLANATION FOR THE 'SOMETHING ELSE', W. PAULI INVENTED THE 'NEUTRINO'.

THE FREE NEUTRON HAS A HALF LIFE OF ABOUT 10 MINUTES.

⑤ A ELECTRON BOUNCES ELASTICALLY OFF A HARD SPHERE OF RADIUS a. WE WISH TO CALCULATE THE RADIATION EMITTED WHEN A BEAM OF ELECTRONS BOUNCE OFF THE SPHERE WITH VARIOUS IMPACT PARAMETERS. WE CAN EXPRESS THE RESULT AS A KIND OF SCATTERING CROSS SECTION FOR EMISSION OF RADIATION.



SUPPOSE THE ELECTRON'S VELOCITY IS MUCH LESS THAN C. THEN FROM PROBLEM ③

$$\frac{dU_\omega}{d\Omega} = \frac{e^2}{4\pi^2 c^3}\left|\hat{n} \times (\bar{v}_1 - \bar{v}_2)\right|^2$$

THE RADIATION IS OBSERVED ALONG DIRECTION $\hat{n}$, WHICH MAKES ANGLE θ TO THE INITIAL DIRECTION OF THE ELECTRON, AND AZIMUTHAL ANGLE ϕ TO THE SCATTERING PLANE SHOWN IN THE SKETCH. WE SUPPOSE ALL ANGLES ϕ ARE EQUALLY LIKELY, AND THE PROBABILITY OF THE ELECTRON HAVING IMPACT PARAMETER b IS $2\pi b\, db$

INTEGRATE OVER b AND ϕ TO OBTAIN $\left\langle \frac{dU_\omega}{d\Omega} \right\rangle$

THE CROSS SECTION FOR EMISSION OF PHOTONS OF ENERGY $\hbar\omega$ IS

$$\frac{d\sigma}{d\Omega\, d\hbar\omega} = \frac{1}{\hbar^2\omega}\left\langle \frac{dU_\omega}{d\Omega} \right\rangle = \frac{a^2}{12\pi}\,\alpha\left(\frac{v}{c}\right)^2 \frac{1}{\hbar\omega}\left(2 + 3\sin^2\theta\right)$$ (SHOW THIS)

THIS LEADS TO $\dfrac{d\sigma}{d\hbar\omega} = \dfrac{4}{3} a^2 \alpha \left(\dfrac{v}{c}\right)^2 \dfrac{1}{\hbar\omega}$ WHERE $\alpha = e^2/\hbar c$

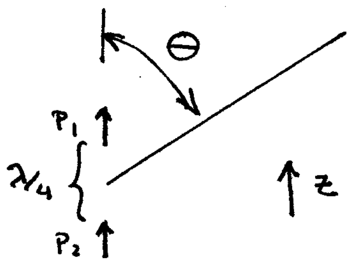
CAN YOU SHOW THAT $\dfrac{d\sigma}{d\Omega\, d\hbar\omega} \equiv \dfrac{d N_{\hbar\omega}}{d\Omega}$ ACTUALLY EQUALS $\dfrac{1}{\hbar^2\omega} \dfrac{dU_\omega}{d\Omega}$?

(6) A PARTICLE OF CHARGE e, REST MASS m_0 IS MOVING IN A CIRCLE IN A CONSTANT MAGNETIC FIELD B. THE CHARGE RADIATES ENERGY AND HENCE DEPARTS FROM THE CIRCLE, SPIRALLING INWARD. SUPPOSE THE CHARGE SPIRALS IN SLOWLY ENOUGH THAT $\bar{a} \perp \bar{v}$ REMAINS A GOOD APPROXIMATION. SHOW THAT $\dfrac{dU}{dt}$ = ENERGY LOSS DUE TO RADIATION, CAN BE INTEGRATED TO GIVE

$$\frac{U}{m_0 c^2} = \coth\left[\frac{2 e^4 B^2 t}{3 c^5 m_0^3} + \text{CONST.}\right]$$

THUS IT TAKES FOREVER TO RADIATE AWAY ITS ENERGY — UNTIL $U \to m_0 c^2$ = REST ENERGY. NOTE THAT YOU MUST GIVE A RELATIVISTIC DERIVATION.

(7) TWO DIPOLE ANTENNAS ARE MOUNTED WITH THEIR AXES COLINEAR AND THEIR CENTERS 1/4 WAVELENGTH APART. ALSO, THE OSCILLATING CURRENTS ARE 90° OUT OF PHASE. FIND THE NET FORCE ON THE ANTENNAS DUE TO THE RADIATION REACTION.

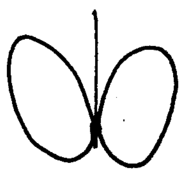


WE SUGGEST YOU FIRST SHOW THAT THE ANGULAR DISTRIBUTION OF THE (TIME AVERAGED) RADIATED POWER IS

$$\frac{dU}{dt\, d\Omega} = \frac{\omega^2}{16\pi c^3} I_0^2 L^2 \sin^2\theta \left(1 - \sin\left(\frac{\pi}{2}\cos\theta\right)\right)$$

(SEE P 191 OF THE NOTES REGARDING A SINGLE DIPOLE ANTENNA)

THIS PATTERN IS ASYMMETRIC, WHICH SHOWS THE UTILITY OF PHASING IN AN ANTENNA ARRAY.



IN ADDITION TO RADIATING ENERGY, THE ANTENNAS RADIATE MOMENTUM. BECAUSE THE ANGULAR DISTRIBUTION IS ASYMMETRIC, THE TOTAL INTEGRAL MOMENTUM RADIATED IS NON-ZERO. SHOW $\bar{F}_R = -\dfrac{d\bar{P}_{\text{RADIATED}}}{dt} = -\dfrac{4\pi\omega^2}{c^4} I_0^2 L^2 \left(1 + \dfrac{6}{\pi^2}\right)$ (MAYBE)

YOU MAY ASK: WHAT DOES THIS RESULT HAVE TO DO WITH THE TERM $\bar{F}_R = -\frac{2}{3}\frac{e^2 \dddot{\bar{x}}}{c^3}$ FOUND IN THE NOTES.

FOR THE OSCILLATORY MOTION IN THIS PROBLEM, THIS AVERAGES TO ZERO.

OUR RESULT GOES LIKE $1/c^4$ NOT $1/c^3$, AND EXCEPT FOR A FACTOR $1/\omega^2$ IN THE RELATION BETWEEN I AND p = DIPOLE MOMENT, WE WOULD HAVE $F_R \sim \frac{\omega^4}{c^4} \sim \frac{\ddddot{x}}{c^4}$

THUS OUR RESULT IS A HIGHER ORDER TERM IN THE SERIES EXPANSION ALLUDED TO IN THE NOTES...

(8) NUCLEAR NUMEROLOGY IN LECTURE 19 WE MENTIONED THE YUKAWA POTENTIAL, $\phi = g\frac{e^{-\mu r}}{r}$ AS THE POTENTIAL OF THE FORCE FIELD WHICH HOLDS THE NUCLEUS TOGETHER. CONSIDER A SINGLE PROTON, OF 'NUCLEAR CHARGE' g. SUPPOSE WE ATTRIBUTE ALL OF THE PROTON'S MASS TO THE ENERGY OF ITS NUCLEAR FORCE FIELD. WHAT IS THE MASS, SUPPOSING THE PROTON IS A SPHERICAL SHELL OF RADIUS a?

HINT: $U = \frac{1}{2}\int \rho\phi \, d\mathrm{vol}$ STILL HOLDS, WHERE ρ = NUCLEAR CHARGE DENSITY.

YOU NEED TO RELATE ρ TO ϕ BY AN APPROPRIATE GENERALIZATION OF POISSON'S EQUATION. YOU SHOULD FIND

$$U = \frac{1}{8\pi}\int [(\bar{\nabla}\phi)^2 + \mu^2\phi^2] \, d\mathrm{vol}.$$

WHAT IS $\frac{M_{PROTON}}{M_{ELECTRON}}$ SUPPOSING BOTH ARE SHELLS OF RADIUS a, AND ALL THE ELECTRON'S MASS IS ELECTROMAGNETIC?

IN THE LIMIT $a \to 0$ YOU SHOULD FIND $\frac{M_p}{M_e} = g^2/e^2$

[CAN YOU GIVE A SIMPLE ARGUMENT WHY THIS MUST HOLD?]

EXPERIMENTALLY, $M_p/M_e = 1836$

IN QUANTUM MECHANICS THE DIMENSIONLESS RATIOS $e^2/\hbar c$ AND $g^2/\hbar c$ PLAY AN IMPORTANT ROLE. OUR MODEL OF THE MASS THEN PREDICTS $g^2/\hbar c \sim 13$ USING $e^2/\hbar c = 1/137$.

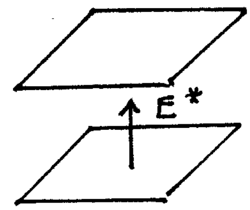
THIS ESTIMATE OF $g^2/\hbar c$ IS ACCURATE TO 10%! IS IT PHYSICS OR NUMEROLOGY?

(9) TRANSFORMATIONS OF FIELD ENERGY

a) FIELDS TIED TO CHARGES. – ILLUSTRATING POINCARÉ'S ARGUMENT.

CONSIDER A PARALLEL PLATE CAPACITOR WHICH HAS BEEN CHARGED SO AS TO HAVE FIELD E^* AS MEASURED IN THE REST FRAME OF THE CAPACITOR. IGNORE EDGE EFFECTS.

THEN $U^* = \frac{1}{8\pi}\int E^{*2}\,d\mathrm{vol}^*$ = FIELD ENERGY IN THE REST FRAME



OF COURSE $\bar{P}^* = 0$ = FIELD MOMENTUM IN THIS FRAME.

CONSIDER THE TWO CASES IN WHICH THIS CAPACITOR IS OBSERVED TO MOVE WITH VELOCITY $\bar{v}$ $\parallel$ AND $\perp$ TO $\bar{E}^*$.

FOR EACH CASE FIND $\bar{E}$ AND $\bar{B}$ VIA A LORENTZ TRANSFORMATION, AND THEN CALCULATE U AND $\bar{P}$ ACCORDING TO POYNTING. YOU SHOULD FIND THAT U AND $\bar{P}$ DO NOT SEEM TO BE TRANSFORMING LIKE PIECES OF A 4-VECTOR. THIS IS ALWAYS THE CASE WHEN THE FIELD LINES ARE TIED TO CHARGES.

BUT THE CAPACITOR IS NOT MECHANICALLY STABLE UNLESS THERE IS SOME ADDITIONAL FORCE TO HOLD THE PLATES APART! FOR EXAMPLE, SUPPOSE THE CAPACITOR IS FILLED WITH A SLAB OF DIELECTRIC. WE TAKE $\epsilon = 1$ FOR SIMPLICITY. THE DIELECTRIC HAS MASS DENSITY ρ^* IN THE REST FRAME. IT IS UNDER COMPRESSIVE STRESS TO KEEP THE PLATES APART, AND IT IS UNDER TENSION TRANSVERSE TO $\bar{E}^*$ TO COUNTERACT THE REPULSION CHARACTERISTIC OF THE MAXWELL STRESS TENSOR OF THE FIELD, T_{ij} AS IN LECTURE 3.

RELATIVISTICALLY WE INTRODUCE THE MECHANICAL STRESS 4-TENSOR FOR THE DIELECTRIC, $P_{\mu\nu}$. ON NOTING THE PHYSICAL SIGNIFICANCE OF THE TERMS IN THE ELECTROMAGNETIC STRESS TENSOR $T_{\mu\nu}$, WE READILY SEE THAT

$$P^*_{\mu\nu} = \left(\begin{array}{c|c} \rho^* c^2 & 0 \\ \hline 0 & P_{ij} \end{array}\right) \qquad \text{WHERE } P_{ij} = -T_{ij}$$

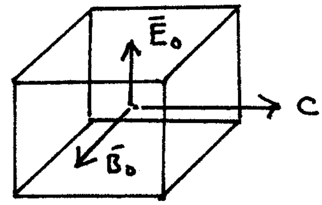
$$\text{THUS} \quad T^*_{\mu\nu} + P^*_{\mu\nu} = \left(\begin{array}{c|c} u^* + \rho^* c^2 & 0 \\ \hline 0 & 0 \end{array}\right)$$

NOW IF WE DEFINE $U^*_{CAP.} = \int (T^*_{00} + P^*_{00})\,d\mathrm{vol}^* = \int (u^* + \rho^* c^2)\,d\mathrm{vol}^* = U^* + M_{DIEL.}c^2$

$\bar{P}_{i\,CAP} = \int (T^*_{0i} + P^*_{0i})\,d\mathrm{vol}^* = 0$ IN REST FRAME

THEN $(U, \bar{P}c)$ WILL INDEED BE A 4 VECTOR, WITH MASS $M_{DIEL.} + \frac{U^*}{c^2}$

b) WAVE FIELDS



NOW WE CONSIDER A RECTANGULAR VOLUME WHICH CONTAINS RADIATION FIELDS $\bar{E}$ AND $\bar{B}$

SAY $\bar{E} = E_0 \hat{x}$, $\bar{B} = B_0 \hat{y}$ SO THAT THE BOX IS MOVING WITH VELOCITY c IN THE $\hat{z}$ DIRECTION.

TRANSFORM TO ANOTHER FRAME SUCH THAT THE ORIGINAL FRAME HAS VELOCITY $v\hat{z}$ ACCORDING TO AN OBSERVER IN THE NEW FRAME.

AGAIN CALCULATE $\bar{E}$ & $\bar{B}$ AND THE EVALUATE U AND $\bar{P}$ ACCORDING TO POYNTING. YOU SHOULD NOW FIND THAT $(U, \bar{P}c)$ TRANSFORM LIKE A 4-VECTOR!

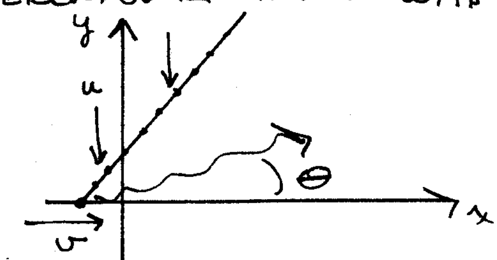
REMARK: BE VERY CAREFUL IN RELATING THE VOLUME OF THE RADIATION IN THE TWO FRAMES.

(10) RADIATION FROM A 'SUPERLUMINAL' SOURCE

CONSIDER A LINE OF CHARGE, $y = \frac{u}{v}x - ut$, THAT MOVES IN THE $-y$ DIRECTION WITH VELOCITY $u \ll c$. HOWEVER, THE SLOPE OF THE LINE IS SUCH THAT THE VELOCITY OF THE INTERCEPT AT $y = 0$ MOVES WITH VELOCITY $v > c$!

(THIS CAN OCCUR IN THE FASTEST ANALOG OSCILLOSCOPE, THE TEKTRONIX 7104.)

SUPPOSE THE REGION $y < 0$ IS A PERFECT CONDUCTOR. SHOW THAT THE TRANSI N RADIATION ASSOCIATED WITH THE DISAPPEARANCE OF THE CHARGE INTO THE CONDUCTOR CAN BE INTERPRETED AS A KIND OF ČERENKOV RADIATION WITH ANGLE $\theta = \cos^{-1} c/v$ TO THE $+x$ AXIS



CONSIDER $\bar{J} = -\hat{y}\, Ne\, \delta(z)\, \delta\left(t - \frac{x}{v} + \frac{y}{u}\right)$

WHAT ABOUT THE IMAGE CURRENT?

CALCULATE $\frac{dU}{d\omega\, d\Omega}$ FOR $u \ll c$. INTEGRATE OVER AZIMUTH, AND CONVERT TO THE PHOTON NUMBER SPECTRUM $\frac{dN_\omega}{d\omega\, d\cos\theta}$

SHOW $dN_\omega \sim \frac{\alpha}{2\pi}(N\lambda)^2 \frac{d\omega}{\omega} \frac{L}{\lambda} \frac{u^2}{c^2}\left(1 + \frac{c^2}{v^2}\right)$ IF $v > c$

FOR THE RADIATION EMITTED WHILE THE CHARGE TRAVERSES DISTANCE L ALONG THE x-AXIS. COMPARE $dN_\omega \sim 2\pi\alpha \frac{d\omega}{\omega} \frac{L}{\lambda} \sin^2\theta_c$ FOR ORDINARY ČERENKOV RADIATION.