PH 206 PROBLEM SET 10

DUE : TUESDAY APRIL 17, 1984 ; MAXIMUM RECORDED SCORE = 70 POINTS

① GRAVITY WAVES . WE HAVE SHOWN THAT WAVES OF THE GRAVITATIONAL TENSOR POTENTIAL,

$\phi_{\mu\nu} = \epsilon_{\mu\nu}\, e^{i(kz-\omega t)}$ ARE TRANSVERSE AND HAVE ONLY TWO DISTINCT POLARIZATIONS. FURTHER, $\phi_{\mu\nu}$ IS SYMMETRIC AND TRACELESS.

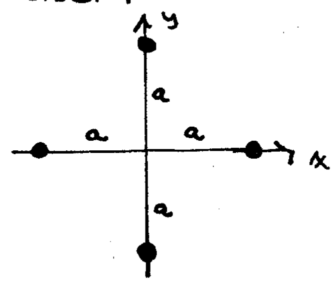
BUT WHAT IS THE SIGNIFICANCE OF THE $\phi_{\mu\nu}$? EINSTEIN TELLS US TO WRITE THE INVARIANT $(\text{LENGTH})^2$ AS

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \qquad \text{INSTEAD OF } ds^2 = dx_\mu dx^\mu .$$

THEN
$$g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} + \phi_{\mu\nu} \quad \text{FOR GRAVITY WAVES IN}$$

'EMPTY SPACE' – FAR FROM ANY STAR OR PLANET. THUS WE MAY SAY THAT $\phi_{\mu\nu}$ IS A WAVE IN THE STRUCTURE OF SPACE AND TIME.

CONSIDER A PLANE WAVE $\epsilon_{\mu\nu}\, e^{i(kz-\omega t)}$ WHICH IS INCIDENT ON 4 EQUAL POINT MASSES IN THE X-Y PLANE. WE HAVE ARGUED THAT CONSERVATION OF MOMENTUM AND ANGULAR MOMENTUM RESTRICT THE SIMPLEST GRAVITIONAL WAVE TO BE DUE TO QUADRUPOLE RADIATION.



USE YOUR 'PHYSICAL INTUITION' TO INDICATE THE ONLY TWO DISTINCT QUADRUPOLE OSCILLATIONS THE WAVE COULD INDUCE ON THE 4-MASS SYSTEM. IDENTIFY THE FORM OF THE POLARIZATION TENSOR $\epsilon_{\mu\nu}$ CORRESPONDING TO THESE TWO OSCILLATIONS.

A 'PRACTICAL' GRAVITY WAVE DETECTOR MIGHT CONSIST OF A MASSIVE SPHERE – SO AS TO BE SENSITIVE TO WAVES FROM ALL DIRECTIONS. SKETCH THE OSCILLATIONS INDUCED IN THE SPHERE BY WAVES OF THE TWO POLARIZATIONS YOU FOUND ABOVE. NOTE THAT A ROTATION BY 45° TAKES ONE CASE INTO THE OTHER (COMPARED TO ROTATION BY 90° FOR DIPOLE RADIATION).

FINALLY, SHOW THAT $\sum_{\text{MASSES}} \Delta s^2$ REMAINS INVARIANT UNDER THE OSCILLATIONS CAUSED BY THE GRAVITY WAVE. (ON THE 4-MASS SYSTEM). MEASURE Δx_μ FROM $x=y=z=0$ AT SOME FIXED TIME $(\Delta t = 0)$.

SHOW THIS IN THE WEAK FIELD LIMIT : LET $\epsilon \ll 1$ BE THE STRENGTH OF A COMPONENT OF $\epsilon_{\mu\nu}$, AND $\delta \ll a$ BE THE AMPLITUDE OF THE INDUCED OSCILLATION. (SUFFICIENT TO SHOW THIS FOR 1 POLARIZATION).

② SUPPOSE WE KNOW THE ANGULAR DISTRIBUTION OF RADIATED POWER $\dfrac{dU^*}{dt^*\,d\Omega^*}$ IN THE INSTANTANEOUS REST FRAME OF A CHARGE (OR GROUP OF CHARGES). THE $*$ INDICATES QUANTITIES MEASURED IN THE REST FRAME. SUPPOSE

$$dU^* = f(\cos\theta^*, \varphi^*)\, d\Omega^*\, dt^*$$

WHERE θ^* AND φ^* ARE MEASURED WITH RESPECT TO A POLAR COORD. SYSTEM WITH AXIS ALONG $\bar{v}$ THE VELOCITY OF THE REST FRAME AS SEEN IN THE LAB FRAME.

SINCE $dP_\mu = (dU, c\,d\bar{P})$ IS A 4-VECTOR, WE KNOW

$$dU^* = \gamma\,(dU - \bar{v}\cdot d\bar{P})$$

HOW ARE dU AND $d\bar{P}$ RELATED?

WHAT ARE THE RELATIONS BETWEEN dt AND dt^*, φ & φ^*, $\cos\theta$ & $\cos\theta^*$?

HINT: $\cos\theta$ IS THE ANGLE OF A LIGHT RAY, SO TRANSFORM A 4-VECTOR SUCH AS $(K, K\sin\theta, 0, K\cos\theta)$ TO SHOW THAT

$$\cos\theta^* = \frac{\cos\theta - \beta}{1-\beta\cos\theta} \qquad (\beta = v/c)$$

COMBINE EVERYTHING TO SHOW $\dfrac{dU}{dt\,d\Omega} = \dfrac{1}{\gamma^4(1-\beta\cos\theta)^3}\, f\!\left(\dfrac{\cos\theta-\beta}{1-\beta\cos\theta}, \varphi\right)$

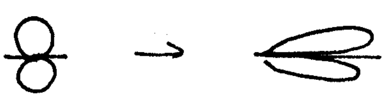
③ USE THE RESULT OF PROBLEM ② TO TRANSFORM THE LARMOR FORMULA, $f = \dfrac{e^2 a^{*2}\sin^2\theta_a^*}{4\pi c^3}$, TO THE LAB FRAME IN THE TWO CASES

a) $\bar{a}^* \parallel \bar{v}$

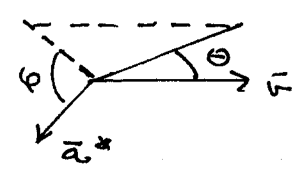
b) $\bar{a}^* \perp \bar{v}$

USE $a^{*2} = -c^4 a_\mu a^\mu$ TO ELIMINATE a^* IN FAVOR OF a IN THE LAB FRAME

ANS: a) $$\frac{dU}{dt\,d\Omega} = \frac{e^2 a^2 \sin^2\theta}{4\pi c^3 (1-\beta\cos\theta)^5}$$



b) $$\frac{dU}{dt\,d\Omega} = \frac{e^2 a^2}{4\pi c^3}\,\frac{(1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\varphi}{(1-\beta\cos\theta)^5}$$



(4) a) SUPPOSE THE ACCELERATION OF A CHARGE e IS ENTIRELY DUE TO EXTERNAL FIELDS $\bar{E}$ AND $\bar{B}$. AS DISCUSSED IN LECTURE 20, THE CHARGE RADIATES ENERGY AND MOMENTUM WITH THE 4-VECTOR

$$dP_\mu = -\frac{2e^2c}{3} a_\nu a^\nu dx_\mu$$

THE PARTICLE OBEYS $\bar{F} = m\bar{a}$ IN THE RELATIVISTIC FORM (LECTURE 18):

$$\frac{dp_\mu}{ds} = f_\mu = F_{\mu\nu} u^\nu \qquad (p_\mu = \text{PARTICLE'S 4-MOMENTUM}):$$

USE THESE FACTS TO SHOW THAT IN AN ARBITRARY INERTIAL FRAME

$$\frac{dU}{dt} = \frac{2e^4\gamma^2}{3M_0^2c^3}\left[\left(\bar{E} + \frac{\bar{v}}{c}\times\bar{B}\right)^2 - \left(\bar{E}\cdot\frac{\bar{v}}{c}\right)^2\right] \qquad (M_0 = \text{REST MASS})$$

b) RECONSIDER a) FROM ANOTHER POINT OF VIEW.

$$\frac{dU}{dt} = \frac{dU^*}{dt^*} = \frac{2}{3}\frac{e^2 a^{*2}}{c^3} \qquad \text{AS } \frac{dU}{dt} \text{ IS A LORENTZ INVARIANT.}$$

IT THEN SUFFICES TO RELATE a^* TO F_{LAB} = FORCE IN LAB FRAME.

SHOW $$\frac{dU}{dt} = \frac{2}{3}\frac{e^2}{M_0^2c^3}\begin{cases} F^2 & \text{IF } \bar{a} \parallel \bar{v} \\ \gamma^2 F^2 & \text{IF } \bar{a} \perp \bar{v}\end{cases}$$

c) THE MAXIMUM LABORATORY ELECTRIC FIELD THAT CAN BE APPLIED TO A MOVING CHARGE IS ABOUT 10^8 VOLTS/METER.

THE MAXIMUM STATIC MAGNETIC FIELD SUITABLE FOR DEFLECTING CHARGED PARTICLES IS ABOUT 100,000 GAUSS.

FOR A HIGHLY RELATIVISTIC ELECTRON ($\gamma \gg 1$) CALCULATE THE RADIATED ENERGY PER METER FOR $\bar{E}$ + AND $\parallel$ TO $\bar{v}$, AND $\bar{B} \perp$ TO $\bar{v}$. $\gamma \sim 10^5$ FOR ELECTRONS PRODUCED AT THE STANFORD LINEAR ACCELERATOR CENTER.

(5) A RELATIVISTIC PARTICLE OF CHARGE e, REST MASS M_0 PASSES BY A FIXED CHARGE CENTER Ze SUCH THAT b IS THE DISTANCE OF CLOSEST APPROACH. WHAT IS THE TOTAL ENERGY RADIATED, $U = \int \frac{dU}{dt} dt$, BY THE CHARGE ASSUMING IT IS HARDLY DEFLECTED AT ALL?

ANS: $$U = \frac{\pi Z^2 e^6}{12 c^3 M_0^2 b^3 v}\,\frac{4 - (v/c)^2}{\left(1-(v/c)^2\right)^2}$$

REMARK: AS A CHECK YOU MAY WISH TO MAKE A QUICK ESTIMATE IN THE SPIRIT OF OUR SHORT-CUT DERIVATION OF RUTHERFORD SCATTERING IN Ph 205. I FOUND $U \sim \frac{2}{3}\frac{Z^2 e^6 \gamma^2}{M_0^2 c^3 b^3 v}$ THIS WAY.

⑥ Charge e_1 with mass m_1 passes by charge e_2 with mass m_2, such that the relative velocity $v \ll c$. However v is still so large that we may suppose the particles move along straight lines. Show that the angular distribution of the emitted radiation is

$$\frac{dU}{d\Omega} = \int \frac{dU}{dt\,d\Omega}\,dt = \frac{e_1^2 e_2^2}{32\,c^3 v\, b^3}\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2 \left(4 - n_x^2 - n_y^2\right)$$

3 ?

supposing $\bar{v} = v\,\hat{x}$ and the particles lie in the x-y plane.

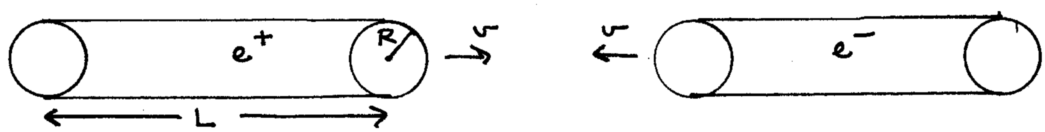
Note that since $v \ll c$ the radiation is well described by the dipole Larmor formula:

$$\frac{dU}{dt\,d\Omega} = \frac{(\hat{n} \times \ddot{\bar{p}})^2}{4\pi c^3} \qquad \text{(Lecture 16)}$$

Recall that if $e_1/m_1 = e_2/m_2$ we expect no dipole radiation!

Finally integrate over $d\Omega$ to obtain the result of problem ⑤ when $e_2 = Z e_1$, $m_2 \to \infty$ and $v/c \ll 1$.

⑦ Colliding Bunches A device is being constructed at Stanford to bring two 'bunches' of electrons and positrons into head-on collision.



Each bunch consists of N particles of charge $+e$ or $-e$. It has radius R and length $L \gg R$. The velocity $v \sim c$ so that $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \gg 1$

a). Estimate the momentum kick given to an electron as it passes thru the positron bunch. By comparing ΔP with P, note that all electrons will be bent so as to cross the axis at the same point. The colliding bunches act like a lens! Show that the focal length of this lens effect is

$$f \approx \frac{\gamma m_0 c^2 R^2}{2 N e^2} = \frac{\gamma R^2}{2 N r_0} \qquad \text{where } r_0 = \frac{e^2}{m_0 c^2} = \text{classical electron radius}$$

(use a thin lens approximation)

b) Due to the deflection the particles emit radiation. Show that for a particle at radius R = edge of bunch, the total radiated energy is approximately (i.e use the quick method suggested at bottom of p. 3)

$$\frac{dU}{U} = \frac{16}{3}\,\frac{\gamma N^2}{R^2 L}\,\frac{e^6}{(m_0 c^2)^2} = \frac{16}{3}\,\frac{\gamma N^2 r_0^3}{R^2 L}$$

The proposed parameters of the Stanford device are

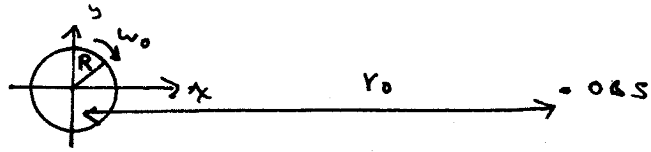
$N \simeq 10^{10}$ particles, $L \sim 1\,\text{mm}$, $R \sim 10^{-4}\,\text{cm}$, $\gamma \sim 10^5$

What is $\frac{dU}{U}$ in this case?

(8) A CHARGE e MOVES IN A CIRCLE OF RADIUS R ABOUT THE ORIGIN IN THE X-Y PLANE.

$x = R \sin \omega_0 t$

$y = R \cos \omega_0 t$



AN OBSERVER IS AT $(x, y) = (r_0, 0)$ WHERE $r_0 \gg R$

THE ELECTRIC FIELD SEEN BY THE OBSERVER IS, IN THE DIPOLE APPROXIMATION

$$\bar{E}_{RAD} = \frac{([\ddot{\bar{P}}] \times \hat{n}) \times \hat{n}}{c^2 r} \sim -\frac{e}{c^2 r_0} \frac{d^2 y(t')}{dt^2} \hat{y}$$

WHERE $t' = t - \frac{r(t')}{c}$ = RETARDED TIME.

EVALUATE $\ddot{y}(t')$ TO SHOW THIS IS $R\omega_0^2 \frac{\cos \omega_0 t' - \beta}{(1 - \beta \cos \omega_0 t')^3}$

THIS IS BIG ONLY IF $\omega_0 t' \sim 2n\pi$. FOR THE PULSE OF RADIATION AROUND $t' = 0$, ELIMINATE t' IN FAVOR OF $T \equiv t - r_0/c$

TO SHOW $E_y(T) \sim \frac{1 - 4\gamma^6 \omega_0^2 T^2}{1 + 12 \gamma^6 \omega_0^2 T^2}$ FOR $\beta \to 1$ AND $T \sim 0$

FIND THE FREQUENCY SPECTRUM OF THIS PULSE. I.e. IF $E(T) = \int_{-\infty}^{\infty} E_\omega e^{-i\omega t} d\omega$

SHOW $E_\omega \sim e^{-\frac{\omega}{2\omega_c}}$ WHERE $\omega_c = \sqrt{3}\, \gamma^3 \omega_0$ AND $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

NOTE THAT THE PULSE INTENSITY HAS ANALYSIS $U = \int_0^\infty U_\omega d\omega$ WITH $U_\omega \sim E_\omega^2 \sim e^{-\omega/\omega_c}$.

THIS APPROXIMATION BREAKS DOWN AT LARGE T, AND SO WE MIS-ESTIMATE THE LOW FREQUENCY PART OF THE SPECTRUM. BUT THIS PROVIDES A GOOD UNDERSTANDING OF THE HIGH FREQUENCY TAIL TO THE SYNCHROTRON RADIATION SPECTRUM. 'EXACT' CALCULATION GIVES $U_\omega \sim \sqrt{\omega}\, e^{-\omega/\omega_c}$ WITH $\omega_c = 3/2\, \gamma^3 \omega_0$. PHYS. REV. 75, 1912 (1949)

(9) A PARTICLE OF CHARGE e_1, MASS M HAS VELOCITY v_0 WHEN FAR FROM A FIXED CHARGE e_2 AT THE ORIGIN. CHARGE 1 IS INITIALLY MOVING DIRECTLY TOWARDS CHARGE 2 - IT MAKES A 'HEAD-ON' COLLISION. IF $v_0 \ll c$ SHOW THAT THE TOTAL ENERGY RADIATED IN THE COLLISION IS

$$\Delta U = \frac{16}{45} \frac{e_1}{e_2} \left(\frac{v_0}{c}\right)^3 \left(\frac{1}{2} M v_0^2\right)$$

NOTE THAT $\frac{\Delta U}{U_0} \sim \left(\frac{v}{c}\right)^3$ AS SEEN IN PROBLEM (6), SET 8.

HINT: WITH THE HELP OF Ph 205 YOU MIGHT SHOW

$$\Delta U = \frac{4}{3} \frac{e_1^2}{M^2 c^3} \sqrt{\frac{M}{2}} \int_{r_{MIN}}^{\infty} \left|\frac{dV}{dr}\right|^2 \frac{dr}{\sqrt{V_{MIN} - V(r)}} \qquad \left[V(r) = \frac{e_1 e_2}{r}\right]$$

AN EVEN QUICKER DERIVATION CAN BE GIVEN BY EMPHASIZING $\int m\, dv$.